A few approaches for finding the Area under Cevians

Chow Ling Hin (HKCCCU Logos Academy)
Chung Yu Hin (HKCCCU Logos Academy, Class MS1T)
Tam Kai Him (HKCCCU Logos Academy, Class MS1T)
Yeung Kwan Chi (HKCCCU Logos Academy, Class MS1V)

Problem for Investigation

ΔABC is an equilateral triangle. D, E and F are points on AB, BC and CA respectively such that $AD : DB = BE : EC = CF : FA = 1 : 2$. AE, BF and CD are called **Cevians** of ΔABC. Find the area bounded by the three cevians (i.e. the area of ΔGHI). (Figure 1)

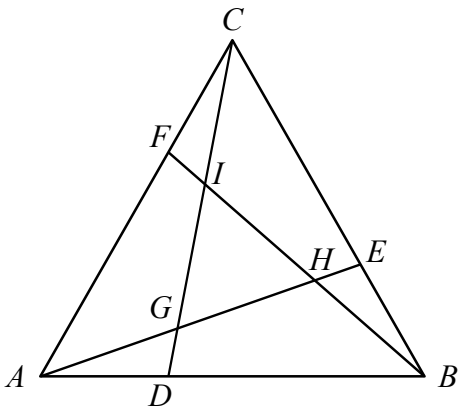

Figure 1

Method 1 (Chung Yu Hin) – Area Ratio

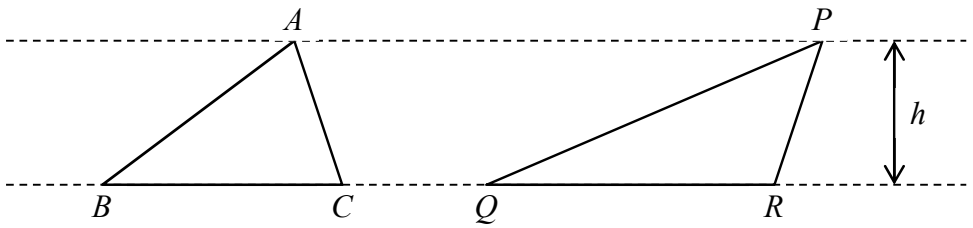

Figure 2

THEOREM

When two triangles have the same height, the ratio of their areas is equivalent to the ratio of their bases. In other words, let's consider ΔABC and ΔPQR, where both triangles share the same height. BC and QR represent the bases of ΔABC and ΔPQR respectively (see Figure 2). According to this theorem,

$$\frac{\text{area of } \Delta ABC}{\text{area of } \Delta PQR} = \frac{BC}{QR} \quad \text{..... (*)}$$

Construct the lines AI, BG and CH.

Let the area of ΔADG be x, area of ΔAGI be y and area of ΔGHI be k.

By symmetry, area of ΔADG = area of ΔBEH = area of $\Delta CFI = x$ and area of ΔAGI = area of ΔBHG = area of $\Delta CIH = y$ (see Figure 3).

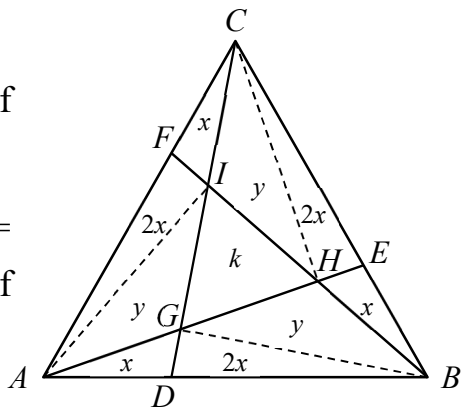

Figure 3

Applying (*) to ΔGAB,

$\because$ $AD : DB = 1 : 2$,

$\therefore$ area of $\Delta GDB = 2x$.

Similarly, area of ΔHEC = area of $\Delta IFA = 2x$.

Applying (*) to ΔIAB,

$\because$ $AD : DB = 1 : 2$,

$\therefore$ area of ΔIAD : area of $\Delta IBD = 1 : 2$.

$$2(x + y) = 2x + y + k$$

$$y = k$$

Consider ΔIAH. By (*), $y = k$ implies $AG = GH$.

Then, consider ΔABH. By (*) again, $AG = GH$ implies $3x = y$.

Note that area of $\Delta ABC = 21x$ and area of $\Delta GHI = 3x$.

$\therefore$ $$\frac{\text{area of } \Delta GHI}{\text{area of } \Delta ABC} = \frac{1}{7}.$$

Method 2 (Chow Ling Hin) – Mechanics

Three particles with weights 2 mg, 1 mg and 4 mg are located at the vertices A, B and C respectively. The center of gravity of A and B is located at D with equivalent weight 3 mg. Note that the center of gravity of particle at C and the resultant weight at D is located along CD. The center of gravity of A and C is located at F with equivalent weight 6 mg. Note that the center of gravity of particle at B and the resultant weight at F is located along BF. From the result above, we can conclude that the center of gravity of A, B and C is located along both CD and BF, i.e. located at I (see Figure 4).

Consider the particle at C and the resultant weight at D. Taking moment at I,

$$4 \text{ mg} \times CI = 3 \text{ mg} \times DI,$$
$$CI : ID = 3 : 4.$$

Similarly, consider the particle at B and the resultant weight at F. Taking moment at I,

$$1 \text{ mg} \times BI = 6 \text{ mg} \times FI,$$
$$BI : IF = 6 : 1.$$

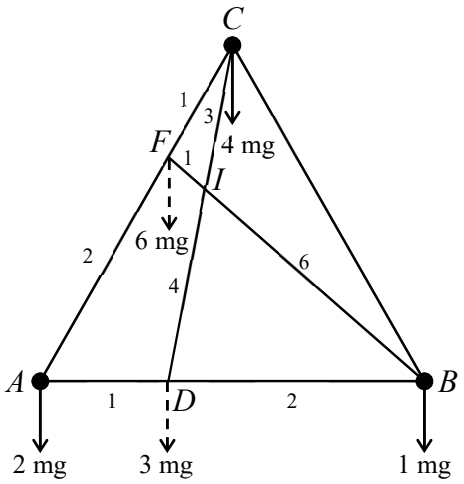

Figure 4

Using symmetry, we can deduce the following ratio in Figure 3.

$$AG : GH : HE = BH : HI : IF = CI : IG : GD = 3 : 3 : 1.$$

By (*), $AG = GH$ implies area of ΔAGI = area of ΔGHI and area of ΔAGB = area of ΔBGH. Using similar approaches, ΔABC can be divided into 7 parts with equal area. Namely, ΔAGB, ΔBGH, ΔBHC, ΔCHI, ΔCIA, ΔAIG and ΔGHI. It is clearly that

$$\text{area of } \Delta GHI = \frac{1}{7} \text{ area of } \Delta ABC.$$

Method 3 (Tam Kai Him) – Trigonometry

Let $AB = BC = CA = 6a$ units. Then $EB = 2a$ units.

Let K be a point on AB such that $AB \perp EK$.

Using trigonometry,

$$BK = 2a \cos 60° \text{ units} = a \text{ units},$$
$$AK = (6a - a) \text{ units} = 5a \text{ units},$$
$$EK = 2a \sin 60° \text{ units} = \sqrt{3}\, a \text{ units}.$$

By Pythagoras' Theorem,

$$AE = \sqrt{5^2 + 3}\, a \text{ units} = \sqrt{28}\, a \text{ units}.$$

From the ratio deduced in method 2,

$$GH : AE = 3 : 7.$$

Therefore, $GH = \frac{6}{\sqrt{7}} a$ units.

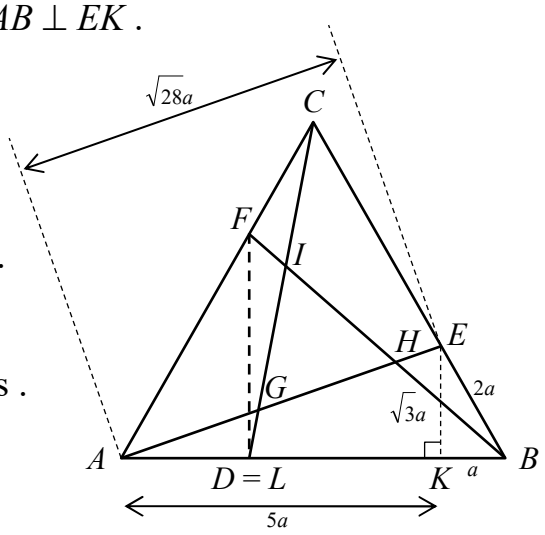

Figure 5

Similarly, $HI = IG = \frac{6}{\sqrt{7}}a$ units .

$\triangle GHI$ is an equilateral triangle and, hence, $\triangle GHI \sim \Delta ABC$.

$$\therefore \quad \frac{\text{area of } \Delta GHI}{\text{area of } \Delta ABC} = \left(\frac{GH}{AB}\right)^2 = \left(\frac{\frac{6}{\sqrt{7}}a}{6a}\right)^2 = \frac{1}{7} .$$

Method 4 (Yeung Kwan Chi) – Areas of Triangles and Trigonometry

Let $AB = BC = CA = 6a$ units .

In ΔABC , perimeter of $\Delta ABC = 18a$ units .

By applying the Heron's Formula,

area of $\Delta ABC = \sqrt{9a(9a-6a)^3}$ sq. units $= 9\sqrt{3}\,a^2$ sq. units .

In ΔABE , $EB = 2a$ units . By Law of Cosines,

$$AE^2 = AB^2 + EB^2 - 2 \times AB \times EB \cos 60°$$

$$AE = \sqrt{(6a)^2 + (2a)^2 - 2(6a)(2a)\cos 60°} \text{ units} = \sqrt{28}\,a \text{ units}$$

$$\sin \angle EAK = \frac{\sqrt{3}}{\sqrt{28}} \quad \text{and} \quad \cos \angle EAK = \frac{5}{\sqrt{28}} .$$

Let L be a point on AB such that $FL \perp AB$.

In ΔFAL , $FL = 4a \sin 60° = 2\sqrt{3}\,a$ units and

$AL = 4a \cos 60° = 2a$ units $= AD$.

Therefore, $D = L$ (as shown in Figure 5).

In ΔFBD , $\sin \angle FBD = \frac{2\sqrt{3}}{\sqrt{28}} = \frac{\sqrt{3}}{\sqrt{7}}$ and

$$\cos \angle FBD = \frac{4}{\sqrt{28}} = \frac{2}{\sqrt{7}} .$$

In ΔABH , applying Law of Sines,

$$\frac{HB}{\sin \angle HAB} = \frac{HA}{\sin \angle HBA}$$

$$\frac{HB}{\frac{\sqrt{3}}{\sqrt{28}}} = \frac{HA}{\frac{\sqrt{3}}{\sqrt{7}}}$$

$$2HB = HA$$

and

$$\frac{HB}{\sin \angle HAB} = \frac{AB}{\sin \angle AHB}$$

$$= \frac{AB}{\sin(180° - \angle HBA - \angle HAB)}$$

$$= \frac{AB}{\sin\angle HBA\cos\angle HAB + \cos\angle HBA\sin\angle HAB}$$

$$\frac{HA}{\frac{\sqrt{3}}{\sqrt{7}}} = \frac{6a}{\frac{\sqrt{3}}{\sqrt{7}}\times\frac{5}{\sqrt{28}} + \frac{2}{\sqrt{7}}\times\frac{\sqrt{3}}{\sqrt{28}}}$$

$$= \frac{6a}{\frac{5\sqrt{3}+2\sqrt{3}}{14}} = \frac{6a}{\frac{7\sqrt{3}}{14}}$$

$$HA = \frac{12a}{\sqrt{7}} \text{ units}$$

$$HB = \frac{1}{2}HA = \frac{6a}{\sqrt{7}} \text{ units}$$

Since $GA = HB$,

$$GH = \left(\frac{12a}{\sqrt{7}} - \frac{6a}{\sqrt{7}}\right) \text{units} = \frac{6a}{\sqrt{7}} \text{ units}.$$

Similarly, $HI = IG = \frac{6a}{\sqrt{7}}$ units. ΔGHI is an equilateral triangle.

In ΔGHI,

$$\text{area of } \Delta GHI = \frac{1}{2}\left(\frac{6}{\sqrt{7}}a\right)^2 \sin 60° \text{ sq. units} = \frac{9\sqrt{3}}{7}a^2 \text{ sq. units}.$$

Therefore, it is clear that $\frac{\text{area of } \Delta GHI}{\text{area of } \Delta ABC} = \frac{1}{7}$.

Further Investigations

The problem can be investigated further in the following directions.

1. The cevian is expressed as the ratio $1 : n$, where $n \neq 1$ and $n \neq 2$.
2. ΔABC is not equilateral.

Reference

傅海倫（2004）。物理原理在數學中的應用。《數學傳播》28 卷 1 期，63 – 64。
傅海倫（2002）。一個數學問題的開放設計。《數學教育》第 15 期，78 – 79。

Author's e-mail: chowlh@logosacademy.edu.hk