A Funny Number Puzzle of Happy New Year 2024

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Introduction

At the beginning of 2024, I received many greeting messages from my teachers, students and friends. The first message that I received came from Prof. Law Huk Yuen who suggested me to try a number puzzle of 2024 (See [1] for instance). More precisely, the puzzle required us to craft an equation of 2024 using the consecutive digits 1 to 9 by employing only basic operations such as addition, subtraction, multiplication and division along with parentheses. In particular, each digit can only be used once. So, if we blindly do it by trial and error, it is not an easy one. Nearly at the same time, my secondary school Mathematics teacher, Principal Lung Tak Yee, shared with us in the WhatsApp group a new year greeting message which was presented in the following format:

Happy New Year 2024

$(20 + 24) + (20 + 24)(20 + 24) + (20 + 24)$

And Looking Forward to 2025

$1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 + 8^3 + 9^3$

Here, we see two beautiful equations as follows:

$$(20 + 24) + (20 + 24)(20 + 24) + (20 + 24) = 2024$$
$$1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 + 8^3 + 9^3 = 2025$$

A quick question in mind was, how did we come up with these wonderful equations?

While we were all attempting to unleash the secrets left behind, our secondary school Chemistry teacher, Principal Chau Chor Shing, joined our discussions happily. He asked, “How about hoping for 2026 and waiting for 2027?” Thankful to all my secondary school teachers and classmates for

bringing so much joy during our discussion in the happy holiday. To me, I really found a chance to promote the study of Mathematics Extended Part M2 to my S4 students because of this new year greeting message. So, I decided to write up this article to record the whole journey as a result.

Strategies for Solving the New Year Number Puzzle

In this section, we are going to introduce two possible strategies, namely, factorization techniques and replacement techniques, for solving the new year number puzzles.

Firstly, regarding the following equation

$$(20 + 24) + (20 + 24)(20 + 24) + (20 + 24) = 2024 ,$$

we may tackle it in the following way.

Notice that $44^2 = 1936 < 2024$ and $45^2 = 2025 > 2024$, we may consider

$$\begin{aligned} 2024 &= 44^2 + 88 \\ &= 44^2 + 2(44) \\ &= (20 + 24)^2 + 2(20 + 24) \\ &= (20 + 24) + (20 + 24)(20 + 24) + (20 + 24) . \quad \text{Done!} \end{aligned}$$

As for another equation $1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 + 8^3 + 9^3 = 2025$, we can take the good opportunity to recap the memory of those students studying the Mathematics Extended Part M2, a typical exercise about proving the following identity by using the Mathematical Induction:

$$\begin{aligned} 1^3 + 2^3 + 3^3 + \ldots + n^3 &= (1 + 2 + 3 + \ldots + n)^2 \\ &= \left(\frac{(1+n)n}{2}\right)^2 \quad \text{for all positive integers } n . \end{aligned}$$

Instead of talking about the details of the proof of the above identity, let's think about how to apply it. Here we go.

Note that $2025 = 45^2 = (5 \times 9)^2 = \left(\frac{10\times 9}{2}\right)^2 = \left(\frac{(1+9)\times 9}{2}\right)^2$, we have

$$2025 = 1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 + 8^3 + 9^3 . \quad \text{Done!}$$

As $2024 = 2025 - 1 = 2025 - 1^3$, we get

$$2024 = 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 + 8^3 + 9^3 .$$

Thus, we can also greet the others by saying "Happy $2^3 + 3^3 + 4^3 + \ldots + 9^3$."

Other than Principal Lung's sharing of greeting messages, I also made a random attempt to solve the number puzzle. Here is a record of my path of thinking:

$$\begin{aligned} 2024 &= 88 \times 23 \\ &= (8 + 80) \times (20 + 3) \\ &= (8 + 5 \times 16) \times (4 \times 5 + 3) \\ &= (8 + 5 \times 2^4) \times (3 \times 7 + 2) \quad \text{.....(*)} \end{aligned}$$

For (*), we hope to have digits 1, 3, 6, 7 and 9 in the second factor if we fixed the digits used in the first factor. So, I employed the "replacement techniques" to achieve the goal:

$$2024 = (8 + 5 \times 2^4) \times [3 \times (1 + 6) + 2] \quad \text{.....(\#)}$$

Now for (#), the trouble is that the digits 7 and 9 are still missing. Worst still, the digits "2" has been repeated in use. To fix this problem, I employed the "replacement technique" in an appropriate manner again. That is, rewrite (#) as follows:

$$2024 = (8 + 5 \times 2^4) \times [3 \times (1 + 6) + 9 - 7]$$

Or similarly, we get $2024 = (8 + 5 \times (9 - 7)^4) \times [3 \times (1 + 6) + 2]$.

No doubt, these solutions may not be perfect as the operators involve exponentiation instead of only the 4 basic arithmetic operators.

So, let us get back to the original number puzzle shared by Prof. Law. Principal Lung has made two another recommendation in the discussion with us:

- $2 \times (3 + 4 \times 5) \times (6 \times 7 - 8 + 9 - 1) = 2024$ .....(1)
- $1 \times 2 - 3 + (4 + 5 + 6) \times (7 + 8) \times 9 = 2024$ .....(2)

To me, I love the one in (2) more because the digits are naturally arranged in sequential order as well. But then the same question arises again. How did Principal Lung come up with these equations? Let us make a conjecture about his work, which could be a very interesting exercise.

For (1), note that $2024 = 2 \times 23 \times 44$. Then we consider

$$\begin{aligned} 2024 &= 2(3 + 20)(42 + 2) \\ &= 2 \times (3 + 4 \times 5)(6 \times 7 + 2) \\ &= 2 \times (3 + 4 \times 5)(6 \times 7 + 1 + 1) && \leftarrow \text{“1” is repeated} \\ &= 2 \times (3 + 4 \times 5)(6 \times 7 + 9 - 8 + 1) && \leftarrow \text{replacement techniques employed} \\ &= 2 \times (3 + 4 \times 5)(6 \times 7 - 8 + 9 + 1) . \end{aligned}$$

For (2), note that $2024 = 2025 - 1 = 45^2 - 1 = 3^2 \times 15^2 - 1$. We have

$$\begin{aligned} 2024 &= 9 \times 15 \times 15 - 1 \\ &= 9 \times (10 + 5) \times (7 + 8) - 1 && \leftarrow \text{replacement techniques employed} \\ &= 9 \times (4 + 6 + 5) \times (7 + 8) - 1 && \leftarrow \text{still missing digits 1 , 2 and 3} \\ &= 9 \times (4 + 6 + 5) \times (7 + 8) + 2 \times 1 - 3 \\ &= 1 \times 2 - 3 + (4 + 6 + 5) \times (7 + 8) \times 9 && \leftarrow \text{by rearranging the terms} \end{aligned}$$

Done! After further verification with Principal Lung, the above conjecture was confirmed as an exact matching of his thinking path.

Indeed, once we master the factorization techniques and replacement techniques mentioned in the above, it becomes much more easy to solve the number puzzle even if we are restricted to use the basic arithmetic operators and no repetition of digits can be used. For instance, here comes a record of my another random attempt:

$$\begin{aligned} 2024 &= 4 \times 22 \times 23 \\ &= 4 \times (2 \times 11) \times (20 + 3) \\ &= 8 \times (7 + 4) \times (2 \times 10 + 3) && \leftarrow \text{want to get rid of the digit “0”} \\ &= 8 \times (7 + 4) \times [2 \times (1 + 9) + 3] && \leftarrow \text{replacement techniques employed, but still missing the digits “6” and “5”} \\ &= 8 \times (7 + 4) \times [2 \times (1 + 9) + 3] \times (6 - 5) && \text{.....(3)} \end{aligned}$$

It’s worth noting that, however, (3) can be rewritten as many other similar forms such as $2024 = 8 \times (7 + 4) \times [2 \times (6 - 5 + 9) + 3] \times 1$. Therefore, we also see that the digit “1” is a tricky number as it can be placed in many different positions as a multiplier, and it can also be replaced in many different ways via the operation of subtraction.

To this end, let us consider another special countdown (using digits 0 – 9 instead) suggested by Fermat's Library (See [2] for instance) and make a conjecture about how this can be worked out:

$$2024 = (10 + (9 + 8 \times 7) \times 6) \times 5 + 4 \times 3 \times 2 \times 1 \quad \text{.....(4)}$$

Possible workout of (4):

Note that
$$\begin{aligned} 2024 &= 2000 + 24 \\ &= 2000 + 4! \\ &= 5(400) + 4 \times 3 \times 2 \times 1 \\ &= 5(10 + 390) + 4 \times 3 \times 2 \times 1 \\ &= 5(10 + 65 \times 6) + 4 \times 3 \times 2 \times 1 \\ &= 5(10 + (8 \times 7 + 9) \times 6) + 4 \times 3 \times 2 \times 1 \end{aligned}$$

So, another way to greet "Happy 2024" is as follows:

"Happy $(10 + (9 + 8 \times 7) \times 6) \times 5 + 4 \times 3 \times 2 \times 1$".

Up till this moment, the readers should be amazed of many interesting ideas of solving the number puzzle mentioned already. Yet, none of the above workouts employ the use of the operator of division. Let us make another somewhat more special way to solve the number puzzle involving "division":

$$2024 = 2^3 \times 11 \times 23 = \frac{4 \times 22 \times 23}{6} \times 6 = \frac{22 \times 23 \times 24}{6} = \frac{(24)(23)(22)}{3 \times 2 \times 1}$$

$$= \frac{(24)(23)(22) \times 21!}{(3!) \times (21!)} = \frac{24!}{3! \times (24-3)!} \quad \leftarrow \text{a result with factorial}$$

Of course, such result may have a defect that the number "24" should be forbidden in use as it is not a single digit number. But it is beautiful enough to create an expression involving the use of factorial. In particular, as $\frac{24!}{3! \times (24-3)!} = C_3^{24} = C_3^{4!}$, we can also greet the others by saying "Happy new year $C_3^{4!}$" if we feel like to do it in this manner. What a fascinating method to link up the concept of combinatorics!

Regarding the solution of "Happy $C_3^{4!}$" presented here, I suddenly come up with a lot of other interesting stories to tell my students studying Mathematics

Extended Module M2. Actually, $C_3^{4!}$ is really a magic number as it can be linked to the idea of the sum of triangular numbers. The idea is as follows:

$$\begin{aligned} C_3^{24} &= C_3^{22+2} \\ &= 1 + (1 + 2) + (1 + 2 + 3) + \ldots + (1 + 2 + 3 + \ldots + 22) \quad \text{.....(**)} \end{aligned}$$

Let $T_1 = 1$, $T_2 = 1 + 2$, $T_3 = 1 + 2 + 3$, …, $T_{22} = 1 + 2 + 3 + \ldots + 22$. Then T_i's are known as the first 22 triangular numbers where $i = 1, 2, 3, \ldots, 22$. In other words, C_3^{24} is simply the sum of the first 22 triangular numbers. No doubt, (**) is another very familiar expression to those students studying Mathematics Extended Module M2 because proving the following identity is certainly one of the standard exercises from the topic of Mathematics Induction:

$$1 + (1 + 2) + (1 + 2 + 3) + \ldots + (1 + 2 + 3 + \ldots + n) = C_3^{n+2} = \frac{1}{6}n(n+1)(n+2)$$

To me, I also highly recommend teaching the equivalence of the following identities during the lessons of Mathematics Extended Module M2 so that students could acquire more insights:

$$\begin{aligned} & \frac{1\times 2}{2}+\frac{2\times 3}{2}+\frac{3\times 4}{2}+\ldots+\frac{(n-1)\times n}{2}+\frac{n\times(n+1)}{2} \\ = \quad & C_2^2+C_2^3+C_2^4+\ldots+C_2^n+C_2^{n+1} \\ = \quad & 1 + (1 + 2) + (1 + 2 + 3) + \ldots + (1 + 2 + 3 + \ldots + n) \\ = \quad & 1 \times n + 2 \times (n - 1) + 3 \times (n - 2) + \ldots + n \times 1 \end{aligned}$$

As for the further details about the derivation of proofs for the above equivalence, the readers may refer to the paper written by Y. Kobayashi [3] which is really a very good teaching resource.

To wrap up, I personally love seeing beautiful patterns in Mathematics very much. I deeply believe that it is interesting to discuss the visual proof presented in [4] with students in which a lively animated YouTube video can be watched in learning how to deduce the following formula about the sum of triangular numbers:

$$1 + (1 + 2) + (1 + 2 + 3) + \ldots + (1 + 2 + 3 + \ldots + n) = \frac{1}{6}n(n+1)(n+2)$$

In Response to Principal Chau's Quiz

In the previous section, we have pointed out that factorization techniques and replacement techniques are both useful tools for solving the new year number puzzles. Recalling that Principal Chau has added a quiz during the discussion with us in our secondary school alumni WhatsApp group, we can also use the replacement techniques to complete the mission instantly, assuming that the repetition in the use of digits is allowed.

For instance, by noticing that $2^3 = 8$ and $3^2 = 9$, we have

$$1^3 + (3^2 - 1) + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 + 8^3 + 9^3 = 2025$$
$$\Rightarrow \quad 1^3 + 3^2 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 + 8^3 + 9^3 = 2025 + 1 = 2026$$

Of course, an even simpler and direct solution is as follows:

$$1 + (1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 + 8^3 + 9^3) = 2026$$

But then the drawback is that the number of repeated digits will increase. Here, both the digits 1 and 3 are repeated whereas the former one (obtained by replacement techniques) only consists of the repeated digit 3. Similarly, by using the same trick, many other variants of equation related to 2026 can be displayed as follows:

Variant 1:

Rewrite 1^3 as $(3^1 - 2)$, we get

$$(3^1 - 2) + (2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 + 8^3 + 9^3) = 2025$$
$$3^1 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 + 8^3 + 9^3 = 2027 = 2026 + 1$$

Thus, $3^1 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 + 8^3 + 9^3 - 1 = 2026$

i.e. $3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 + 8^3 + 9^3 - 1 = 2026$, with only the digit 3 being repeated.

Variant 2:

A direct consequence of variant 1 leads to the following equation:

$$2^1 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 + 8^3 + 9^3 = 2026,$$ with both the digits 2 and 3 being repeated, which is less nice than that of variant 1 with more digits being repeated but nicer than that of variant 1 in the sense that less kinds of

operators were involved (i.e. "–" operator is NOT required.)

Thus, the underlying message of the quiz given by Principal Chau to us is insightful. We can simply deliver this kind of number pattern game to our students to play in order to enhance their creativity. As an exercise, the reader may think about how to make use of "replacement techniques" to come up with the following equation in which the exponents of the terms are arranged in ascending order.

$$-4^2 + (1^3 + 2^3 + 3^3 + 5^3 + 6^3 + 7^3 + 8^3 + 9^3) + 3^4 = 2026$$

Concluding Remarks

In this article, it serves as a memorable record about the discussion of number puzzles for the new year 2024 with my teachers and classmates (secondary and university) in our WhatsApp groups. Other than presenting some possible solutions of the number puzzle 2024 for our appreciation, we also suggest some possible strategies (factorization techniques and replacement techniques) to help us solve similar puzzles. More importantly, I would like to bring out two key messages here.

First, not only solving the number puzzles, but also thinking about how other people solve the number puzzles is in fact very interesting and insightful to our learning in Mathematics.

Second, these kinds of number puzzles could be delivered to students to enhance their higher order thinking skills and creativity. Also, the rubrics could be assessed based on the number of operators used, the beauty of the result (e.g. symmetry of the pattern) and the number of digits being repeated in use and so on.

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References

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