Different Methods to solve a Plane Geometry Question

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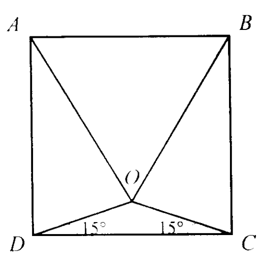
Introduction

The converse of a theorem in plane geometry sometimes provides a good exercise for nurturing high-level thinking in students, even though the proof of the original theorem is straightforward. In some cases, the converse is a much more difficult theorem to prove, e.g. the converse of theorem which asserts that an isosceles triangle has two bisectors of the base angles equal. In some cases, the converse of the theorem can be proved in many ways. In this article the authors will illustrate the latter category by the following question.

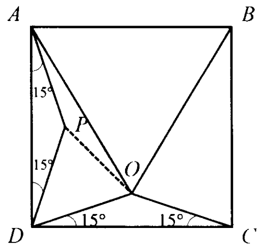
Question

Given that $ABCD$ is a square, and O is a point inside the square such that $\angle ODC = \angle OCD = 15°$. Prove that AOB is an equilateral triangle.



(Note that if O is a point inside the square such that AOB is an equilateral triangle, then it is straight-forward to prove $\angle ODC = \angle OCD = 15°$.)

Method 1



Construct $\Delta PAD \cong \Delta ODC$ inside $ABCD$.

$DO = CO$ (Sides of opp. eq. $\angle$s)

$\therefore$ $AP = PD = DO = OC$

Also $\angle PAD = \angle PDA = \angle ODC = 15°$

$\angle ADC = 90°$ (Property of a square)

$\therefore$ $\angle PDO = 60°$

$\angle DPO = \angle DOP$ (base $\angle$s of isos. Δ)

$\therefore$ $\angle DPO = 60°$ ($\angle$ sum of Δ)

$\angle APD = 150°$ ($\angle$ sum of Δ)

$\therefore$ $\angle APO = 150°$ ($\angle$s at a point)

$\angle PAD = \angle PDA$ (base $\angle$s of isos. Δ)

$\therefore$ $\angle PAD = 15°$ ($\angle$ sum of Δ)

$\angle DAB = 90°$ (Property of square)

$\therefore$ $\angle OAB = 60°$

Similarly, $\angle OBA = 60°$

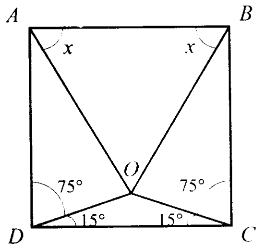
$\therefore$ $\angle AOB = 60°$ ($\angle$ sum of Δ)

$\therefore$ ΔAOB is equilateral.

Method 2

$\angle ADC = \angle BCD = 90°$ (Property of square)

$\angle ADO = \angle BCO = 75°$



$DO = CO$ (sides of opp. eq. $\angle$s)

$AD = BC$ (Property of square)

$\therefore$ $\Delta ADO \cong \Delta BCO$ (SAS)

$\therefore$ $AO = BO$

$\angle AOD = \angle BOC$

Let $\angle OAB = x$, then $\angle OBA = x$ (Base $\angle$s of isos Δ)

$\angle AOB = 180° - 2x$ ($\angle$ sum of Δ)

$\angle DOC = 150°$ ($\angle$ sum of Δ)

$\therefore$ $\angle AOD = x + 15°$ so $\angle ADO = 75°$

If x < 60° then $\angle AOD < \angle ADO = 75°$ $\therefore$ $AD < AO$

Also $\angle ABO < \angle AOB$ $\therefore$ $AO < AB$

i.e. $AD < AB$ which contradicts the fact that $ABCD$ is a square.

$\therefore$ x cannot be smaller than $60°$.

Similarly, x cannot be larger than $60°$

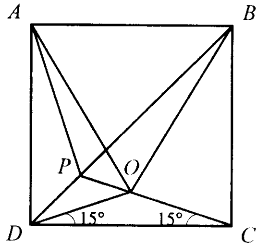
$\therefore$ $x = 60°$

$\therefore$ $\angle ABO = \angle BAO = \angle AOB = 60°$

$\therefore$ ΔABC is equilateral.

Method 3

Produce CO to meet BD at P. Join AP.



$\angle ADP = \angle CDP = 45°$ (Property of square)

$DP = DP$ (common)

$AD = DC$ (Property of square)

$\therefore$ $\Delta ADP \cong \Delta CDP$ (SAS)

$\therefore$ $AP = PC$

$\angle DAP = \angle DCP = 15°$

$\angle DPA = \angle DPC$

$\angle PDO = 30°$

$\angle POD = 30°$ (ext. $\angle$ of Δ)

$\therefore$ $PO = PD$ (Sides of opp. eq. $\angle$s)

$\angle DPO = 120°$ ($\angle$ sum of Δ)

$\therefore$ $\angle DPA = 120°$

$\angle APO = 120°$ ($\angle$s at a point)

$\therefore$ $\Delta APO \cong \Delta CPD$ (SAS)

$\therefore$ $AO = CD$

Similarly $BO = CD$

$\therefore$ $AO = BO = AB = CD$

$\therefore$ ΔAOB is equilateral.

Method 4

Construct an equilateral ΔCDP as shown.

$DO = CO$ (opp. sides of eq. $\angle$s)

$OP = OP$ (common sides)

$DP = CP$ (sides of equil. Δ)

$\therefore$ $\Delta DOP \cong \Delta COP$ (SSS)

$\angle DPO = \angle CPO = 30°$

$\angle ADO = 75° = \angle PDO$

$AD = PD = CD$ (Sides of square)
$DO = DO$ (common)

$\therefore$ $\Delta ADO \cong \Delta PDO$ (SAS)

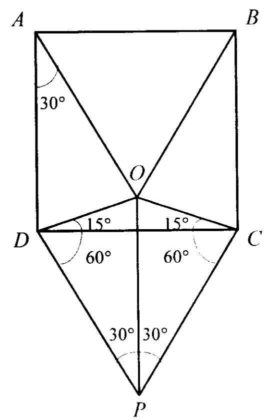
$\therefore$ $\angle DAO = \angle DPO = 30°$

$\therefore$ $\angle BAO = 60°$

Similarly $\angle ABO = 60°$

$\therefore$ $\angle AOB = 60°$ ($\angle$ sum of Δ)

$\therefore$ ΔAOB is equilateral.



Method 5

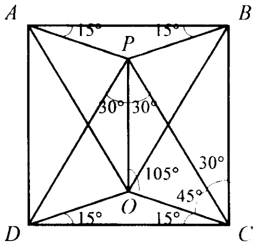
Construct an equilateral ΔCPD as shown.

$\angle PCD = 60°$ ($\angle$ of an equil. Δ)

$\therefore$ $\angle BCP = 30°$ (Property of square)

$\angle BPC = \angle PBC = 75°$ (base $\angle$s of isos. Δ)

$\therefore \quad \Delta ABP \cong \Delta DOC \quad$ (ASA)



$\therefore \quad PB = OC \quad$ (corr. sides)

$PO = PO \quad$ (common)

$\angle BPO = \angle COP = 105°$

$\therefore \quad \Delta PBO \cong \Delta OCP \quad$ (SAS)

$\therefore \quad OB = CP \quad$ (corr. sides)

$OB = CP = CB = AB$

Similarly $OA = DP = DA = AB$

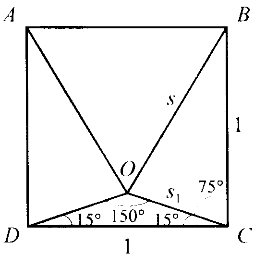
$\therefore \quad \Delta AOB$ is equilateral.

Method 6

Applying sine and cosine laws.
Let $AB = BC = CD = DA = 1$ (Property of square)

$$\frac{1}{\sin 150°} = \frac{s_1}{\sin 15°}$$

$$\therefore \quad s_1 = \frac{1\sin 15°}{\sin 150°} = \frac{\sin 15°}{\sin 30°} = \frac{1}{2\cos 15°}$$



$$\begin{aligned} s^2 &= 1^2 + {s_1}^2 - 2 \times 1 \times s_1 \times \cos 75° \\ &= 1 + \frac{1}{4\cos^2 15°} - 2 \times \frac{1}{2\cos 15°} \times \sin 15° \\ &= 1 + \frac{1}{4\cos^2 15°} - \frac{2\sin 15° \cos 15°}{2\cos^2 15°} \\ &= 1 + \frac{1}{4\cos^2 15°} - \frac{\sin 30°}{2\cos^2 15°} \\ &= 1 \end{aligned}$$

$\therefore \quad s = 1$

Similarly $AO = 1$

$\therefore \quad OB = OA = AB = 1$

$\therefore \quad \Delta AOB$ is equilateral.

Method 7

Construct an equilateral triangle AQB with Q inside the square.
It is straight-forward to prove

$\angle QDC = \angle QCD = 15°$.

Hence, QD, QC coincide OD , OC respectively, i.e. Q and O coincide.

$\therefore$ AOB is same as AQB, which is an equilateral triangle.