Some Applications of Hand-Held Technology in Teaching Mathematics at Upper Secondary Level in Hong Kong

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This article exemplifies the uses of Hand-held Technology (HHT) in teaching and learning mathematics in the current school syllabus, base on two conjectured 3-step teaching algorithms. Students' motivation for learning is greatly enhanced trough it. Yet traditional learning strategies such as making mathematical conjectures with rigorous proofs are still retained.

Some advantages of using HHT (one type of IT resources) in Mathematics teaching and learning are as follows:

(a). More Flexible Pedagogy in Mathematics Education
(b). Interesting students' Explorations in Mathematical Investigations with high motivation

And the characteristics of using HHT are:

(a)' Multi-dimensional ways of teaching/learning : **NUMERICAL / GRAPHICAL / SYMBOLIC (or ALGEBRAIC)** representations [abbreviated as **(NGS) / (NGA)**]
(b)' Significance of Mathematical knowledge through **VISUALIZATION**
E.g. visualization of formulae, theorems, limits and inductive steps

In fact, (a)' is a distinctive feature [1] of HHT.

Facing the five-year strategy on IT (1998), school colleagues need to be well equipped with some simple and flexible teaching algorithm in using IT.

For instead, geometric investigations through Dynamic Geometry software [e.g. Cabri Geometry I & II, Sketchpad].

A three-step teaching algorithm [2], accounting for students' process of learning in Geometry through IT is suggested as follows:

Steps of Implementation	Functions	J. Biggs' Solo Taxonomy model (1982)
1. Continuous graphical displays with versatile functional tools within one single geometrical figure through Dynamic Geometry	Draw students' intuitive attention of discovering intrinsic features being preserved through interactive manipulations	Pre-structural / Uni-structural / Multi-structural Levels
2. Teach / guide students to use their intuitive understanding previously gained in step 1 to formulate hypotheses within their preliminary cognitive bases	Students are motivated to articulate hypotheses concerning those preserved features & relationships in step 1	Multi-structural / Relational Levels
3. Teach / guide students to correlate their procedural / conditional knowledge with propositional ones , implicitly embedded in the hypotheses formulated in step 2	Students confirm / falsify the hypotheses articulated in step 2 by offering rigorous proofs / solutions through generalization of individual cases intuitively explored in step 1	Relational / Extended Abstract Level

Similarly, a similar 3-step algorithm in learning Algebra, Arithmetic, Pre-calculus and Calculus through IT can be as follows:

Steps of Implementation	Functions	J.Biggs' Solo Taxonomy
1'. Graphical / Numerical Articulation of Problems Through Graphing Calculators	Students gain intuitive understanding in Graphical / Numerical perspectives	Pre-structural / Uni-structural / Multi-structural Levels
2'. Guide / teach students how to interpret / conceptualize such Graphical / Numerical information in step 1'	Students are motivated to formulate hypotheses incorporating those Graphical / Numerical data gained from step 1'	Multi-structural / Relational Level
3.' Guide / teach students students to correlate their procedural / conditional knowledge with propositional ones , implicitly embedded in the hypotheses formulated in step 2'	Students confirm / falsify hypotheses articulated in step 2' by paper-pencil computations or computer symbolic algebra (CSA) in some graphing calculators e.g. TI-92 through generalization of numerical cases / graphical interpretation in step 1'	Relational / Extended Abstract Level

For step 1 [or 1'], the tasks are easily accomplished by students of mixed abilities in Dynamic Geometry / graphing calculators. However, the transitions from step 1 to step 2 [or from step 1' to step 2'] and from step 2 to step 3 [or from step 2' to step 3'] require teachers ' further stimulation. Traditional training in their heuristic reasoning [e.g. Polya G. (1962)] needs to be severely involved when launching the steps 2 and 3 [or (2' and 3')].

To vindicate wild application of such teaching algorithms, three examples[3] in S.4-S.5 Additional Mathematics / S.6-S.7 Pure Mathematics are given.

1. Visualization of inductive steps [modified from Gossez (1997)]

Teaching strategies	**Expected students' answers**	**Students' process of learning**
Step 1' : ask students to find out and interpret any special relationship(s) / pattern(s) among numerical integral x- & y-values in the graph in fig. 3 / numerical table in fig.4 by TI-92	Graphical-Numerical representation suffices to show : F (1) = 1 F (2)-f (1)= 2 F (3)- f (2)= 3 And onwards	Uni-structural level of learning is attained with that Graphical-Numerical linkage
Step 2' : ask them : (a) To generalize the relationship(s) / pattern(s) in step 1' [c.f. fig.5] (b) To reinterpret the generalized relationship(s) / pattern(s) through Graphical- Numerical representation in step 1'	(a) Their hypothesis : f (n)–f (n-1) = n (all n ε **N**) (c) ½ (1)(2) = f (1) =1 ½ (2)(3) = f(2) = 1 + 2 go onto: ½ n (n+1) = f(n) = 1 + 2 +…..+ n (all n ε **N**) a hypothesized inductive statement	Multi-structural / Relational level is attained when articulating hypotheses / through generalization
Step 3' : ask them to confirm / falsify that inductive argument by rigorous proof	For inductive step : Assume P (n=k) is true : 1+2+…..+k = k (k+1) / 2 , but f(k+1) = f(k) + (k +1) [c.f. step 2'(a)] So (k+1)[(k+1)+1] / 2 = f (k+1) = [1+2+…..+k] + (k+1) Hence, P (n=k+1) is true	Students ' conceptual understanding of that inductive step can be visualized by the generalized result at step 2' (a) and their proof is done at extended abstract level

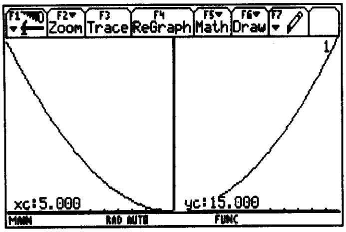
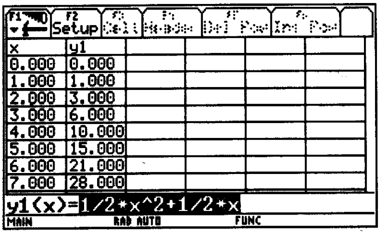


Figure 1:
Graph of y =f(x)= ½ x^2+ ½ x = ½ x (x+1) and with ZoomInt mode in TI-92

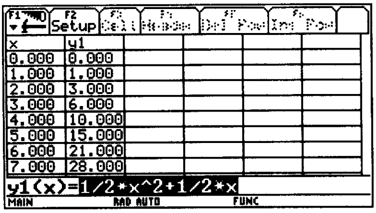


x	y1
0.000	0.000
1.000	1.000
2.000	3.000
3.000	6.000
4.000	10.000
5.000	15.000
6.000	21.000
7.000	28.000

y1(x)=1/2*x^2+1/2*x

Figure 2:
Numerical display of integral x-y-values under f(x) with Table mode in TI-92

Such algorithm can be applied to other inductive statements involving an integral variable.



x	y1
0.000	0.000
1.000	1.000
2.000	3.000
3.000	6.000
4.000	10.000
5.000	15.000
6.000	21.000
7.000	28.000

y1(x)=1/2*x^2+1/2*x

Figure 3:
Use cumSum to check the relationship between f(n) and f(n + 1)

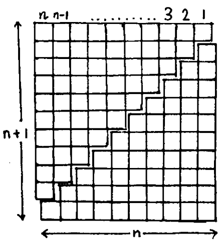


Figure 4:
Geometric proof of P(n): 1 + 2 + … + n = (1/2)n(n + 1) [all n ∈N]

Pure geometrical proof [e.g. fig .6] of inductive arguments requires teachers' extra knowledge. Teachers may not easily formulate some sophisticated ones. However, such algorithm has no such requirement and can effectively enhance students' understanding of the inductive steps through a progressive 3-level process of learning [in Biggs' terminology]. On the other hand, extended abstract level of learning, at which low-ability students cannot be easily attained, is stressed in many school textbooks in Additional Mathematics [e.g. Chow W.K. (1994)].

2. Visualization [4] of complex nth-roots of unity

Teaching strategies	**Students' expected answers**	**Their process of learning**
1. Consider P_1 = cis θ. (a) ROTATE P1 by θ to P_2 and find its coordinates (b) visualize the product of $(\text{cis }\theta)^2$ on Argand plane [c.f. fig. 12 when θ = 30 °, 45 °, 75 °, 120°] (c) articulate the relationship between cis θ and cis 2θ	Students' understanding of the relationship % Cartesian and Polar Co-ordinates done before such exploration [c.f fig. 11] and algebraic multiplication of two complex numbers. Intuitively, they easily realize : $(\text{cis }\theta)^2 = \text{cis } 2\theta$ i.e. algebraic power of cis θ is equivalent to its geometric rotation	Multi-structural level of geometric understanding of algebraic multiplication involves the power of cis θ
2. (a) ask them to explore further the relationship between P_1 & $P_n = \text{cis } n\theta$, n ε **N** (b) ask them to consider : what would be those complex roots (Z) for : $Z^n = 1$, n ε **N** [Hint: use the above rotation mechanism to find ALL roots Z.] (c) ask them to explore relationship among those roots to $Z^n = 1$, n ε **N** [Hint: use the above rotation mechanism]	(a) they can easily conjecture: $(\text{cis }\theta)^n = \text{cis } n\theta$, n ε **N** (b) by the interactive rotation mechanism, they visualize : $Z = \text{cis }(2k\pi / n)$, k = 0 ,1 , ...n-1 . Ongoing values of k (=n , n+1, n+2 ,.....) only repeat the n roots. (c)There are totally n complete n-th roots of unity. Some are power of others. Conjugate of each root is also a root [c.f. n=3 in fig.13 , n=4 in fig.14]	Relational level of the relationship between algebraic power of cis θ and its geometric rotation is attained (a) Another relational level of geometric understanding of algebraic n-th roots of unity is achieved (b) Another relational level of geometric interrelationships among those n-th roots is reached
3. (a) Ask them to use mathematical induction to prove general statement	Traditional works found in many textbooks in (a)-(c).	Interactive geometric-algebraic perspectives of n-th roots of unity can be consolidated at

(with integral variable, n) in 2 (a) (b) Ask them to algebraically verify the general statement in 2 (b) (c) Ask them to use algebraic properties of complex numbers and conjugates to confirm the relationships in 2(c) .		extended abstract level

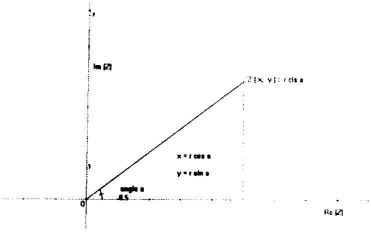


Figure 5

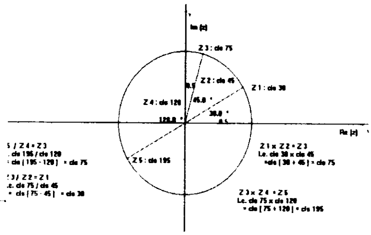


Figure 6

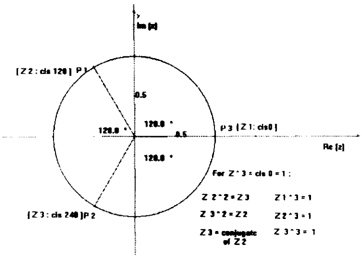


Figure 7

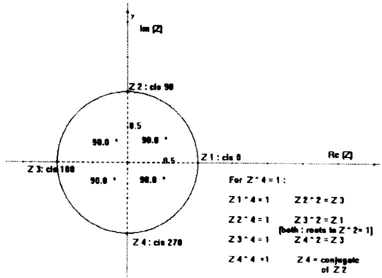


Figure 8

Some students fail to gain much conceptual understanding of complex n-th roots of unity when following teaching guidelines in school texts. It is probably due to lack of geometric-algebraic investigation of the roots that may be attained by the initial levels through HHT.

Furthermore, the above teaching strategy can enhance students' understanding when a polynomial in z being converted into a algebraic product of pairs of factors by considering the corresponding geometrical relationships among complex roots of unity

e.g. $z^{2n} - 1 = \prod_{k=0}^{n-1}\left(z^2 - 2z\cos\frac{2k+1}{2n} + 1\right)$ in Cabri Geometry.

3. Visualization of (non) differentiability of functions

Teaching strategies	**Students' expected answers**	**Their process of learning**
(1) (a) ask students to Graph : y=[sin(x+h)-sinx] / h e.g. y =[sin (x+0.5)–sin x] / 0.5; y=[sin (x+0.1)-sin x] / 0.1; y=[sin (x+0.01)-sin x]/0.01; [in fig.15, Broadwin (1997)] (b) let them to explore [using the Table / Graph form] when h becomes vary small, the resulting function f (x) =[sin(x+h)-sinx] / h will become another function through graphing other guessed ones and comparing numerical y-values in TI-92 (c) Consider other cases : g (x) = [cos(x+h)-cosx] / h; I (x) = [tan(x+h)-tanx] / h	From the trend reviewed by the numerical values table / graphical representation , they can easily guess : (I) sin′x = cos x ; (ii) cos′x = – sin x; (iii) tan ′ x = $\sec^2$x	Even if the numerical-graphical demonstration can be regarded as a strict proof , the tendency of numerical values / resulting graphs [when h tends to be zero] suffice to illustrate the first derivatives at uni- / multi-structural level
2. After discovering the traditional geometrical meaning of derivatives , ask them to conjecture: (i) sin′x = cos x ; (ii) cos′x = – sin x; (iii) tan ′ x = $\sec^2$x	For some particular point x_1 , they can verify the slope of tangent to each function at x_1 is numerically equal to the above corresponding limit [when h tends to be zero]	New geometric/ numerical meaning of the derivative of each function can be linked with traditional one at relational level
3. Based on the first principles, confirm the hypotheses.	Book works in school texts	Symbolic understanding of derivatives can be attained at extended abstract level
1'. (a) Consider f (x) =|x| and f(x) = $x^{2/3}$ ask them to explore when x tends to zero , the limit	(a) Based on numerical-graphical demonstrations , they	(a) They can gain intuitive understanding of

of f(x) and the left- / right-hand limit with geometric / numerical explanations: [$\lim_{h \to 0}$ f (x)- f(x-h) / h] & [$\lim_{h \to 0}$ f (x+h)-f(x) / h] (b) Other explorations for x^n, $x^n \|x\|$, $x^{1/n}$, $x^{n-1/n}$ (n is a positive integer) 2.' Let them conjecture : which function being continuous at a point may not be necessarily differentiable at that point	can gain numerical / geometrical meanings of non-differentiability of functions as left- and right-hand derivatives are unequal [e.g. fig. 16] (b) Some continuous functions are non-differentiable at x=0 for some values of n whilst others are differentiable for all values of n. x^n (n is odd) ,$x^n \|x\|$ (all n ε N) , $x^{1/n}$ (n is odd) are differentiable at 0 [e.g. fig.17 , fig.18] whereas $\|x\|$, $x^{n-1/n}$ (n is odd) are not differentiable at 0 [e.g. fig.19 & fig.20]	differentiability of functions at uni-/multi-structural level (b) At relational level , the concepts of differentiability / non-differentiability of functions can be extended to other functions in numerical-graphical representations At relational level , they will correlate those cases with differentiable / non-differentiable points at 0 whether n is even or odd
3.' Use the first principles to confirm the conjectures	Bookwork / exercises in school texts and examinations	At extended abstract level , they can generalize / categorize those cases with (non-) differentiable points at 0

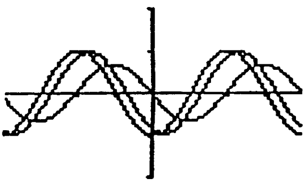


Figure 9

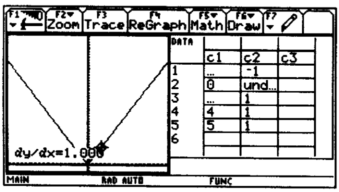


Figure 10

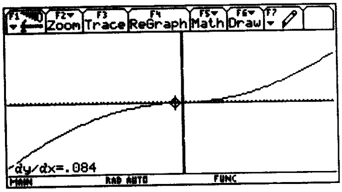


Figure 11: left-hand derivative of x |x| near zero

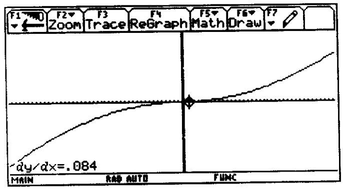


Figure 12 : right –hand derivative of x |x| near zero

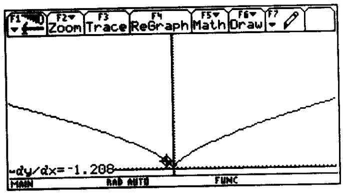


Figure 13: left derivative of $x^{2/3}$ near zero

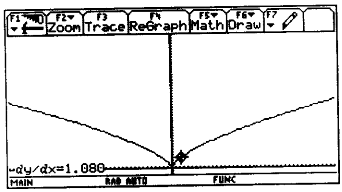


Figure 14 : right derivative of $x^{2/3}$ near zero

Through such numerical-graphical perspective, they can gain intuitive understanding of abstract concepts of derivatives, continuity and differentiability. This will enrich/deepen their learning in the resulting analysis.

This paper is wholeheartedly dedicated to Mr. Raymond Chow Wai Man who envisioned the potential development of HHT in Mathematics Education few years ago. We wish he could recover from Hepatitis B very soon and join our restless struggle for its full utilization.

Special thanks are given to Prof. Siu Man Keung , Mr. Henry Chan Yip-Cheung in the Dept. of Mathematics at the University of Hong Kong and school colleagues named Mr. Hilary Lai Hoi Yuen in Hong Kong and Mr. Ling Yi-guo in mainland China for their continuous, insightful stimulation and comments through e-mail. Meanwhile, I am greatly indebted to Prof. Edward Laughbaum (in the Dept. of Mathematics at Ohio-State University in U.S.A.) on the debatable definitions of HHT.

Notes:

1. In this article, HHT is widely defined as those technological resources that can be manipulated by free skilled hands e.g. graphing calculators (i.e. *handheld calculators*), the combination of such PC software as WinPlot (Graphical),

Excel (Mainly Numerical) and Derive (Algebraic) stored in *laptops*, with explorations in collecting data from the real world through TI-CBL / TI-CBR in the field of Mathematics / Science / Engineering / Business. Comparatively speaking, some Mathematics educators / schoolteachers like to narrowly regard HHT as graphing calculators. Strictly speaking, complete (GNA) / (GNS) representation is a sufficient but *not necessary* condition of HHT. For such hand-held technology as TI-CBL /-CBR can have mere Numerical (N) representation.

2. Such students' learning process models are *mere conjectures* in Educational Psychology. Future in-depth case (classroom-/group-based) studies need to be carried out to vindicate its functioning in-groups of mixed-ability students'. Its modification may be involved in various settings.
3. Throughout this paper, TI-92 and Cabri II (stored in PC version) are used for demonstration. Other models of graphing calculators and Dynamic Geometry software can also be utilized with similar operational routines.
4. Such example works only in cases of complex numbers with *unity* modulus / magnitude. Complete geometrical resolution of roots to any complex equations like $z^n = r \text{ cis } \theta$ ($r > 1$ or $0 < r < 1$) is infeasible in Cabri Geometry. For any geometrical transformations done in its functional tools cannot be equivalent to algebraic operations of taking nth algebraic roots of complex numbers with *irrational* modulus / magnitude.

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