Two Interesting Activities for S1 Mathematics Lessons

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1. Introduction

Recently, I have attended a series of 3 Algebra workshops organized by "Hong Kong Association of Mathematics Education" (HKAME). As one of the participants of the workshops, there are two very interesting activities that caught my attention very much. One of them is about solving algebraic equation related to a given story. Another one is a game about divisibility of numbers. In this article, I would like to share with the readers some of my afterthoughts about these two funny activities which are suitable to S1 Mathematics lessons.

2. Solving Algebraic Equations by Using "Reverse Thinking Methodology"

In this section, we will discuss how to solve an algebra problem with a story similar to the one depicted during the workshop. The details of the story are described as below:

> Mark has a bag consisting of x candies and he would like to distribute some candies to Angel, Celia and Emily on the way back home from work. When he met Angel in a road towards his home, he asked Angel to get one half of the number of candies from his bag first and then get one more candy. Later on, when he met Celia in another location, he asked Celia to do it in the same way. That is, Celia will get one half of the number of candies from his bag first and then get one more candy. Lastly, when he met Emily in another location nearby his home, he repeated the same instruction again such that Emily will also get one half of the number of candies from his bag first and then get one more candy. When Mark finally reached his home, he found that there were

only 2 candies left inside his bag. The question is, what is the total number of candies that were inside his bag originally?

To solve the problem from the above story, we can formulate it as a linear algebraic equation with a special structure like the one shown below:

$$\frac{\frac{\frac{x}{2}-1}{2}-1}{2}-1=2 \quad \Leftrightarrow \quad \frac{\frac{\frac{x-2}{2}-2}{2}-2}{2}=2 \text{ ---------------------- (*)}$$

Now, there are two different ways that we can consider for solving (*).

- **<u>Method 1</u>**

Try to simplify the L.H.S. of (*) step by step from the topmost layer.
That is,

$$\frac{\frac{\frac{x-2}{2}-2}{2}-2}{2}=2$$

$$\frac{\left[\frac{\left(\frac{x-2-4}{2}\right)}{2}-2\right]}{2}=2$$

$$\frac{\left[\frac{\left(\frac{x-6}{2}\right)-4}{2}\right]}{2}=2$$

$$\frac{\left[\frac{\left(\frac{x-6-8}{2}\right)}{2}\right]}{2}=2$$

$$\frac{x-14}{8}=2$$

$x=16+14=30.$
So, there are altogether 30 candies that were inside Mark’s bag originally.

- **Method 2**

We can simplify L.H.S. of (*) step by step from the outermost layer. That is,

$$\frac{\frac{\frac{x-2}{2}-2}{2}-2}{2}=2$$

$$\frac{\frac{x-2}{2}-2}{2}=2(2)+2=6 \text{ ------------------- (1)}$$

$$\frac{x-2}{2}=6(2)+2=14 \text{ ------------------- (2)}$$

$$x=14(2)+2=30 \text{ ------------------------ (3)}$$

So, there are altogether 30 candies that were inside Mark's bag originally.

During the workshop, some participants pointed out that the solution can be obtained by using "reverse thinking methodology" very easily. More precisely, the short-cut to get the answer is given by:

$\{[2(2)+2]\times 2+2\}\times 2+2$

$=\{6\times 2+2\}\times 2+2=\{12+2\}\times 2+2=28+2=30.$

Yes, that's really a nice way in solving the problem. In fact, the idea of such a short-cut is exactly coming from the <u>Method 2</u> as shown in the above. To further explain this phenomenon, let us look at the problem again with the help of Figure 1 shown below.

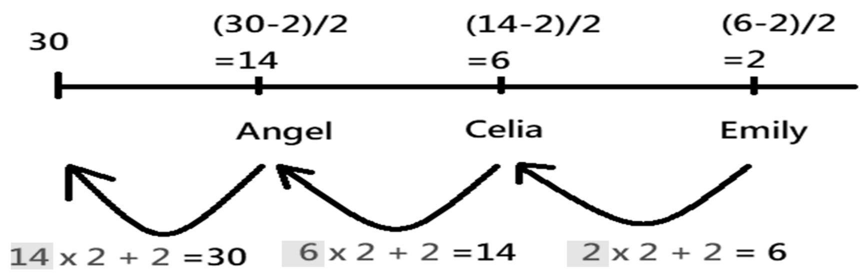

Figure 1: Timeline representation of the distribution of candies

Suppose that we look at Figure 1 from the left to the right. We can see the remaining number of candies after each distribution clearly:

30 (original number in bag) → 14 → 6 → 2 (remaining number after all the distribution)

However, we can also look at Figure 1 from the right to the left. In this way, it will illustrate the backward process of restoring the original number of candies in the bag step by step if we begin with the final remaining number of candies after all the distribution being done.

2 (last remaining number in bag) → 6 → 14 → 30 (original number in bag)

<u>*Remarks</u>

For the first restoration of 2 → 6, it corresponds to step (1) in Method 2 which gives the remaining number of candies after the distribution to Celia.

For the second restoration of 6 → 14, it corresponds to step (2) in Method 2 which gives the remaining number of candies after the distribution to Angel.

For the third restoration of 14 → 30, it corresponds to step (3) in Method 2 which gives the original number of candies in the bag before any distribution.

Thus, the problem depicted in the above story can be solved by repeating the process of "times 2, plus 2" for 3 times easily. In fact, such an efficient technique of reversing the order for solving the linear equation is very common in dealing with Mathematics Olympiad problems. (See pp.14 from [1] for instance.)

At this moment, the readers may think that this activity of solving an algebra problem based on a given story has already come to an end. Not really! A very exciting moment comes suddenly. There was an inspiring question raised by Mr. Bobby Poon (Chairperson of HKAME) during the workshop. His question was as follows: What if the right-hand side of the linear equation incurred by the given story is 3 instead of 2? Do we need to re-do all the calculations again? Oh, what a nice way to check students' understanding and provoke their higher-order thinking skills. Regarding Bobby's inspiring question, l would like to suggest the following lesson activity as a follow-up for introducing the technique of using "reverse thinking methodology" to solve linear equations.

First of all, assuming that the short-cut technique for solving the original linear equation related to the story has been solved. That is, $x = 30$ is the solution of $\frac{\frac{\frac{x-2}{2}-2}{2}-2}{2} = 2$.

Then, we can ask students what will be the new equation that we need to solve if the final number of candies that were remaining inside the bag is 3 instead of 2. The answer is of course the following equation:

$$\frac{\frac{\frac{x-2}{2}-2}{2}-2}{2} = 3 \text{ -------------- (\#)}$$

By using the technique of "reverse thinking methodologies", the solution of (#) is given by:

$$x = \{[(3)(2) + 2](2) + 2\}(2) + 2 = 38$$

Next, we can ask students to change the value of the right-hand side to be 4, 5 and 6 etc...

For solving $\frac{\frac{\frac{x-2}{2}-2}{2}-2}{2} = 4$ ----------- (##), we get:

$$x = \{[(4)(2) + 2](2) + 2\}(2) + 2 = 46 \;;$$

For solving $\frac{\frac{\frac{x-2}{2}-2}{2}-2}{2} = 5$ ----------- (###), we get:

$$x = \{[(5)(2) + 2](2) + 2\}(2) + 2 = 54 \;;$$

For solving $\frac{\frac{\frac{x-2}{2}-2}{2}-2}{2} = 6$ ----------- (####), we get:

$$x = \{[(6)(2) + 2](2) + 2\}(2) + 2 = 62 \;.$$

As a result, students should be able to create a sequence of numbers such as $\{30, 38, 46, 54, 62\}$ which corresponds to different values of x for different cases. Now, the teacher may ask students whether they can discover the special feature of such a sequence of numbers. It is expected that students can point out that the sequence has a common difference of 8. (i.e., An arithmetic sequence can be formed if the right-hand side is incremented by 1 every time.) In this way, students will not only be able to unlock the secret that the answer of Bobby's query is simply obtained by $30 + 8 = 38$, but also solve many other similar problems in which only the right-hand side of the linear system is incremented by a certain value.

Next, the teacher can grasp a chance to strengthen students' understanding by asking why is so? How to prove it mathematically? The following possible explanations can be done by either student themselves or to be demonstrated by teachers.

- Explanation 1:

Consider the right-hand sides of the linear equation concerned differ by one. Let x_1 and x_2 be the solution of $\frac{\frac{\frac{x-2}{2}-2}{2}-2}{2} = k$ and $\frac{\frac{\frac{x-2}{2}-2}{2}-2}{2} = k+1$ respectively, where k is a constant.

Then, by applying the technique of "reverse thinking methodology", we have:

$$x_1 = [(k \times 2 + 2) \times 2 + 2] \times 2 + 2 \ .$$

Now, $x_2 = [((k+1) \times 2 + 2) \times 2 + 2] \times 2 + 2$

$$= [(k \times 2 + \mathbf{2} + 2) \times 2 + 2] \times 2 + 2$$

$$= [(k \times 2 + 2 + \mathbf{2}) \times 2 + 2] \times 2 + 2$$

$$= [(k \times 2 + 2) \times 2 + \mathbf{2} \times 2 + 2] \times 2 + 2$$

$$= [(k \times 2 + 2) \times 2 + 2 + \mathbf{4}] \times 2 + 2$$

$$= [(k \times 2 + 2) \times 2 + 2] \times 2 + \mathbf{4} \times 2 + 2$$

$$= [(k \times 2 + 2) \times 2 + 2] \times 2 + \mathbf{8} + 2$$

$$= [(k \times 2 + 2) \times 2 + 2] \times 2 + 2 + \mathbf{8}$$

$$= x_1 + 8$$

So, the new solution x_2 can simply be obtained by adding **8** to the previous solution x_1.

- <u>Explanation 2:</u>

Note that the two algebraic equations presented in Explanation 1 can be rewritten in the following forms:

$$\frac{1}{2}\left\{\frac{1}{2}\left[\frac{1}{2}(x-2)-2\right]-2\right\} = k \text{ -------------- (\$)}$$

$$\frac{1}{2}\left\{\frac{1}{2}\left[\frac{1}{2}(x-2)-2\right]-2\right\} = k+1 \text{ ------- (\$\$)}$$

For solving ($),

$$\frac{1}{8}(x-2) + \left(\frac{1}{4}\right)(-2) + \left(\frac{1}{2}\right)(-2) = k \ .$$

$$\frac{1}{8}(x-2) = k - N \text{ , where } N = \left(\frac{1}{4}\right)(-2) + \left(\frac{1}{2}\right)(-2) \ .$$

So, $x = 8(k-N) + 2$.

For solving ($$), we simply need to solve $\frac{1}{8}(x-2) = (k+1) - N$.

Thus, $x - 2 = 8[(k + \mathbf{1}) - N] = 8[(k-N) + \mathbf{1}] = 8(k-N) + \mathbf{8}$.

i.e. $x = [8(k-N)+2] + \mathbf{8}$ which is just the same as adding **8** to the solution obtained in solving equation ($).

After going through all the activities described in the above, students should be able to master the technique of "reverse thinking methodology" for solving linear equations with a special structure based on the given story. The following exercises could be delivered to students for further practices.

- <u>Exercise 1:</u>

Question: Solve $\frac{\frac{\frac{x-2}{2}-2}{2}-2}{2} = 2023$ by using the solution of $\frac{\frac{\frac{x-2}{2}-2}{2}-2}{2} = 2.$

Solution: Note that $2023 - 2 = 2021$.

The required solution is given by $\mathbf{30} + 8(2021) = 16198$ where $\mathbf{30}$ is the solution obtained for $\frac{\frac{\frac{x-2}{2}-2}{2}-2}{2} = 2$.

- <u>Exercise 2:</u>

Question:

(a) Solve $\frac{\frac{\frac{x-2}{3}-2}{4}-2}{5} = 2$ by using the technique of "reverse thinking methodology".

(b) Design a story which requires solving the given equation.

Solution:

(a) $x = \{[2(5) + 2](4) + 2\}(3) + 2 = 152$.

(b) This question is open-ended which aims to enhance the creativity of students. Here comes a sample answer.

> Shannon was given a string of x cm for doing an experiment. Firstly, she cut the string into two parts and then removed the small part of 2 cm long. For the remaining part, she tried to cut the string into 3 identical pieces. Next, she took one of these 3 identical pieces and cut it again into two parts and then removed the small part of 2 cm long again. For the remaining part obtained in this stage, she tried to cut the string into 4 identical pieces. Finally, she took one of these 4 identical pieces and cut it again into two parts and then removed the small part of 2 cm long. For the remaining part obtained in this stage, she tried to cut the string into 5 identical pieces. From then she discovered that each of these 5 identical

pieces has the length of 2 cm too. She asked her groupmate Melanie to find the original length of the whole string.

3. Games About Divisibility of Numbers

In S1 mathematics lessons, the first chapter is usually about the basic computation techniques such as the divisibility of numbers. For instance, in order to determine whether a given number is divisible by 3, students will learn the rule by checking whether the sum of all digits of the given number is divisible by 3 or not. (See [2] and [3] for instance)

However, if we teach students purely the rule by direct instructions, the whole lesson becomes very boring. That's the reasons why many teachers will arrange the lesson in the form of playing a game. Basically, there are two popular approaches for the arrangement of the games about divisibility of numbers.

- Distribute some numbered cards to students. Ask students to follow a rule (written as a hint) to check whether the given number is divisible by 3, and the process will be repeated for a few numbers of trials. The first one who can finish all the tasks about divisibility tests will become the winner. (See [4] for instance)

- Distribute some numbered cards to students. Ask students to check whether the numbers written in the cards received are divisible by 3 or not, and then guess the rules of divisibility by 3. Next, students are divided into groups and take turn to get a numbered card so as to check the divisibility of the number being shown on the card chosen. The one who gets a correct answer can keep the chosen card. After a certain number of rounds, the one who keeps the most numbered cards will be regarded as the winner. (See [5] for instance)

While gamification lessons can be effective for enhancing students' learning motivation, the games described in the above can only lead to one and only one learning outcome. That is, students will be able to learn the rule of divisibility by 3. During the Algebra workshop, we were happy to be exposed to another more interesting game about the divisibility of numbers by 3. The operations of this game are similar to the one depicted below.

> A set of 15 cards numbered 1, 2, 3, ..., 15 were distributed to a group of 15 people. Each participant of the game will hold his/her card such that all the other people can see the content of others' cards clearly. Then, the participants have to try forming a single number by using his own card and two more cards from the other two participants such that the number so formed is divisible by 3. The first group who can complete the mission will become the winner.

In order to play the above-mentioned game, we may use brute force method or simply try to add up the 3 numbers from any 3 numbered chosen cards and then check whether the result is divisible by 3 or not. However, the participants may also discover the following tricky ways eventually.

(1) Divide the set of 15 numbers {1, 2, 3, . . . ,14, 15} into 3 groups:
Group 1: {1, 4, 7, 10, 13} → all elements in the form of $3n-2$, where n = 1, 2, 3, ...
Group 2: {2, 5, 8, 11, 14} → all elements in the form of $3n-1$, where n = 1, 2, 3, ...
Group 3: {3, 5, 9, 12, 15} → all elements in the form of $3n$, where n = 1, 2, 3, ...

(2) Next, there are two simple methods to form a number that is divisible by 3.
(I) Select any 3 numbers within the same group. By appending the numbers together from the chosen cards to form a single number (e.g. 11310, 2511, 1239, etc...), the result must be divisible by 3.
(II) Select one number from each group. By appending the 3 numbers together from the chosen cards to form a single number (e.g. 147, 2811, 31215), the result must be divisible by 3.

Now, as a round up, the teacher may explain the reason why the above magic is workable.

Suppose all the 3 numbers were chosen from Group 1. We have:
sum of all digits of the number formed
$= (3n_1 - 2) + (3n_2 - 2) + (3n_3 - 2)$ for some natural numbers n_1, n_2 and n_3
$= 3(n_1 + n_2 + n_3) - 6 = 3(n_1 + n_2 + n_3 - 2)$, which is divisible by 3.

Suppose all the 3 numbers were chosen from Group 2. We then have:
sum of all digits of the number formed
$= (3n_1 - 1) + (3n_2 - 1) + (3n_3 - 1)$ for some natural numbers n_1, n_2 and n_3
$= 3(n_1 + n_2 + n_3) - 3 = 3(n_1 + n_2 + n_3 - 1)$, which is also divisible by 3.

Similarly, if all the 3 numbers were chosen from Group 3, we have:
sum of all digits of the number formed
$= 3n_1 + 3n_2 + 3n_3$ for some natural numbers n_1, n_2 and n_3
$= 3(n_1 + n_2 + n_3)$, which is clearly divisible by 3.

Finally, if we select only one number from each group, then
sum of all digits of the number formed
$= (3n_1 - 2) + (3n_2 - 1) + 3n_3$ for some natural numbers n_1, n_2 and n_3
$= 3(n_1 + n_2 + n_3) - 3 = 3(n_1 + n_2 + n_3 - 1)$, which is once again divisible by 3.

Anyway, with a slight variation of a game like this to be arranged in our S1 mathematics lessons, I believe that the students can obtain more fruitful insights at the end.

4. Concluding Remarks

Instead of teaching S1 students about how to solve linear equations routinely, it is recommended to expose them to more real-life stories like the one introduced by the Algebra workshop so that students will feel that mathematics is really very useful to them. The Exercise 2(b) suggested in Section 2 exactly fits this purpose. In particular, it can also train students' communication skills and language skills during their learning of mathematics as they are required to present stories involving mathematical concepts themselves. As for the idea of "reverse thinking methodology", it is a very useful skill for students to deal with problems coming from Mathematics Olympiad. The technique of "reverse thinking methodology" is very simple in its operations and I believe that students' learning motivation can also be boosted up after learning such a wonderful technique. Last but not least, gamification can be a useful strategy to enhance the learning motivation if we can have a more thoughtful design in the logistics.

5. Acknowledgement

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6. References

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