G is the Centre of the Napoleon Triangle

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Introduction

The Napoleon Triangle for any triangle is formed by the three centres of the equilateral triangles "constructed" on the sides of the triangle. The Napoleon Triangle Theorem states that the Napoleon Triangle is equilateral. A proof of the Theorem is presented and as a shocking fact that the centre of the Napoleon triangle is the centroid of the given triangle.

Construction of equilateral triangles and the Fermat Point

For any given triangle *ABC*, let *XBC*, *YCA* and *ZAB* be the equilateral triangles constructed outside of ΔABC and let P_1, P_2 and P_3 be the centres of the equilateral triangles *XBC*, *YCA* and *ZAB* respectively (Figure 1). We are going to prove that the Napoleon Triangle $\Delta P_1P_2P_3$ is equilateral.

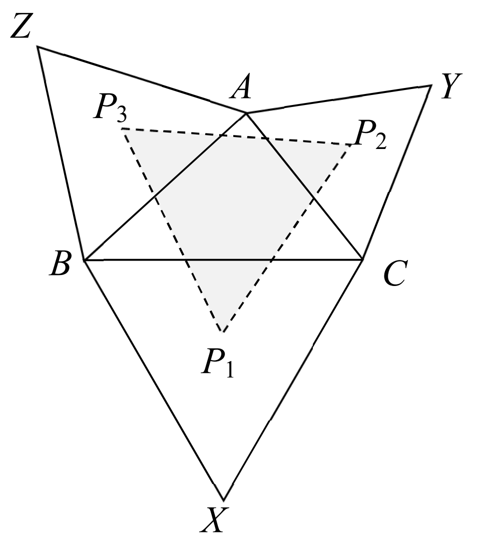


Figure 1 – The Napoleon Triangle

Proposition 1. *XA*, *YB*, *ZC* are equal and concurrent. Furthermore, the angle between any two of them is 60°. (Figure 2)

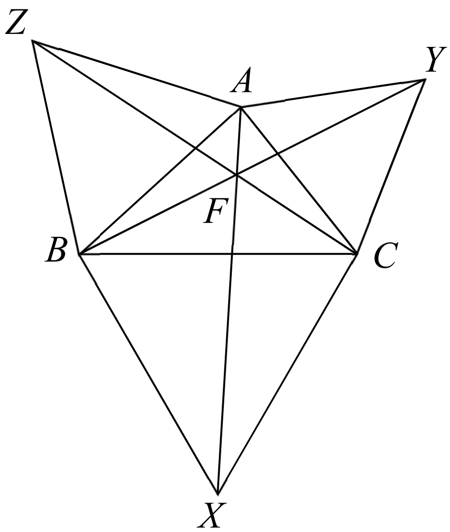


Figure 2 – The Fermat Point on 3 Lines

Proof of Proposition 1. We first construct *XA* and *YB* and let *F* be their point of intersection. From the equilateral triangles *XBC* and *YCA*, we have *XC* = *BC*, *AC* = *YC* and $\angle XCA = 60° + \angle BCA = \angle BCY$. It follows that ΔXAC and ΔBYC are congruent, and hence the corresponding sides *XA* and *BY* are equal. Similarly, by considering congruent triangles ΔXAB and ΔCZB, it follows that *XA* = *CZ*. Hence *XA* = *YB* = *ZC*.

For the angles between the lines and concurrency, we make use of some circle theorems. We first observe that from the pair of congruent triangles ΔXAC and ΔBYC, we have equal corresponding angles

$$\angle XAC = \angle BYC \quad \text{and} \quad \angle AXC = \angle YBC$$

It follows that from the converse of angles in the same segment, the quadrilaterals *FAYC* and *FCXB* are cyclic (Figure 3). Hence, all the angles at circumference $\angle AFY = \angle YFC = \angle CFX = \angle XFB = 60°$.

So from the angles at point *F*, $\angle AFB = 360 - 4 \times 60° = 120°$. So the opposite angles $\angle AFB$ and $\angle AZB$ (= 60°) are supplementary, and hence *FBZA* is also cyclic. Thus, $\angle BFZ = \angle ZFA = 60°$ as angles in the same segments. In particular, we have $\angle ZFC = 180°$ and hence F also lies on the line segment CZ. The proposition is proved.

We denote F as the Fermat Point of the triangle ABC. It lies on AX, BY, CZ, and circumcircles of the three constructed equilateral triangles. The Fermat Point has many geometric properties, for example, it is the point where sum of distances from the three vertices A, B and C is minimum (Coexeter & Greitzer, 1967).

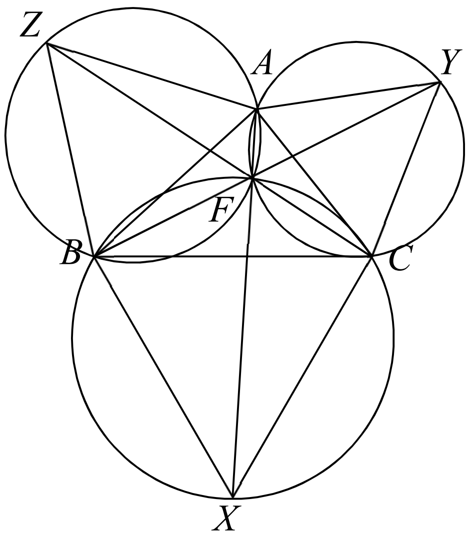
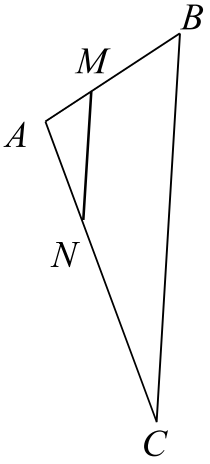


Figure 3 – The Fermat Point on 3 Circles

Proposition 2. $GP_1 = GP_2 = GP_3$ and $\angle P_1GP_2 = \angle P_2GP_3 = \angle P_3GP_1 = 120°$. (Figure 5)

For the proof of Proposition 2, we would need the following lemma.

Lemma. In any triangle ABC, if M and N are the points on the side AB and AC respectively such that $AM : MB = AN : NC = 1 : r$, for some positive number r, then $MN : BC = 1 : (1 + r)$ and $MN \,//\, BC$ (Figure 4).



If $\dfrac{AM}{MB} = \dfrac{AN}{NC} = \dfrac{1}{r}$,

then $\dfrac{MN}{BC} = \dfrac{1}{1+r}$ and $MN \parallel BC$.

Figure 4 – the Lemma (generalized Mid-point Theorem)

This is a generalized version of the Mid-point Theorem; note that when $r = 1$, the lemma is the Mid-point Theorem. The lemma can be proved by considering the pair of similar triangles ΔAMN and ΔABC. It can also be proved easily using vectors.

Proof of Proposition 2. Note that G and P_1 are both centroids of triangles. Let A' denote the mid-point of BC, then G and P_1 lie on the medians AA' and XA' respectively. By the property of centroids, we have

$$A'G : GA = A'P_1 : P_1X = 1 : 2.$$

By the lemma for $r = 2$, we conclude that

$$GP_1 : AX = 1 : 3 \quad \text{and} \quad GP_1 \parallel AX.$$

The properties for P_2 and P_3 can be derived using the same method. Hence,

$$AX = 3GP_1, \quad BY = 3GP_2, \quad CZ = 3GP_3, \quad \text{and}$$
$$AX \parallel GP_1. \quad BY \parallel GP_2. \quad CZ \parallel GP_3.$$

From Proposition 1, we have $AX = BY = CZ$, it follows that

$$GP_1 = GP_2 = GP_3.$$

It also follows from Proposition 1 that the directed lines AX , BY , CZ are 120° differ from each other (figure 3), hence

$$\angle P_1GP_2 = \angle P_2GP_3 = \angle P_3GP_1 = 120°.$$

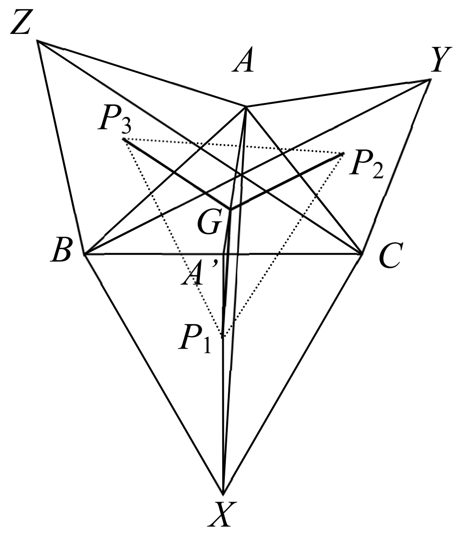


Figure 5

Theorem. The centroid G of a triangle is the centre of the corresponding Napoleon Triangle.

Proof of Theorem. By Proposition 2, $GP_1 = GP_2 = GP_3$ and $\angle P_1GP_2 = \angle P_2GP_3 = \angle P_3GP_1 = 120°$. It follows by congruent triangle condition SAS that $\Delta P_1GP_2 \cong \Delta P_2GP_3 \cong \Delta P_3GP_1$.

Hence $P_1P_2 = P_2P_3 = P_3P_1$, i.e. the Napoleon Triangle $P_1P_2P_3$ is equilateral. And G is the centre of it, as $GP_1 = GP_2 = GP_3$.

Remarks:

1. In an equilateral triangle, the four centres: the centroid, the incentre, the circumcentre and the orthocentre all coincide. We simply call them the *centre* of the equilateral triangle.

2. In terms of geometric transformation, the results of Proposition 1 are just rotations of the segment AX with centre C through 60° clockwise and centre B through 60° anticlockwise to BY and CZ respectively. It then follows by a reduction of AX, BY and CZ from the midpoints by a scale factor of $\frac{1}{3}$ to GP_1, GP_2, GP_3 respectively.

3. The results also holds for the "Inner" Napoleon Triangle, for definition please see (Coexeter & Greitzer 1967).

References

Coexeter, H.S.M. & Greitzer S.L. (1967). *Geometry revisited.* Mathematical Association of America.

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