一題多解？一題多問！

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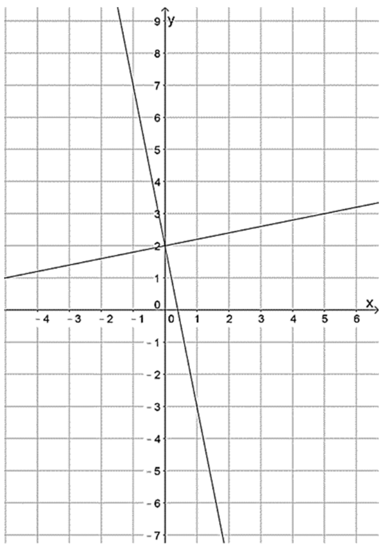
作為數學老師，對於一題多解一定不陌生。

要讓學生學會從不同角度分析數學問題，一個簡單的方法是以不同方式去呈現同一道問題。

筆者教授國際文憑大學預科課程（International Baccalaureate Diploma Programme，簡稱 IBDP）多年，一直都要教授向量這課題[1]。當學生有了直線的向量方程知識後，我會讓學生討論如何找出兩條在平面上的直線的角平分線。而每一次，一進入課室，我先請學生分組，然後讓每組自由選擇以下其中一張工作紙作討論。

《工作紙一》

Find the equations of the bisectors of the angles between the two given lines.



1 早年「數學」（普通水平）（Mathematics Standard Level）也包含向量這課題，現在則只有「數學：分析與嘗試」（高級水平）（Mathematics: analysis and approaches - Higher Level）和「數學：應用與理解」（高級水平）（Mathematics: applications and interpretation - Higher Level）才涵蓋。

《工作紙二》

The line L_1 with slope $\frac{1}{5}$ passes through (0 , 2).

The line L_2 passes through (–1 , 7) and (1 , –3).
Find the equations of the bisectors of the angles between the two given lines.

《工作紙三》

The line L_1 with direction $\begin{pmatrix} 5 \\ 1 \end{pmatrix}$ cuts the y-axis at 2.

The line L_2 passes through (–1 , 7) and is parallel to $\begin{pmatrix} -1 \\ 5 \end{pmatrix}$.

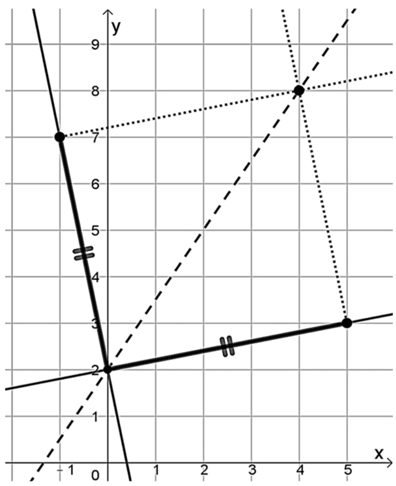
Find the equations of the bisectors of the angles between the two given lines.

不難想像，當學生正在學習向量，而《工作紙三》也提及向量，拿著這張工作紙的學生，都會從向量出發解決問題〔見附錄一〕。

拿著《工作紙二》的學生，有的仍然選擇以向量入手，因為大家正在學習向量嘛！也有學生或因題目只提及坐標、又或較熟悉坐標幾何，所以只用過往的知識去解決問題〔見附錄二〕。

《工作紙一》只提供一幅圖，學生的想像空間變得無限，他們解題的方法，也更見創意。

因為看見，所以有學生能快速知道角平分線與兩線各成 45 度，減省了一大堆計算，便畫出了角平分線：



最快速解決問題的同學，不一定數學能力很高。遇過有學生拿著工作紙向燈對摺，便立即「畫」出角平分線！這個創意解題方法，在疫情期間見不到。網課時、又或間中的在校面授課，題目也只在網上發放。沒有一紙在手，學生也欠缺了誘發創意的空間。不單題目的敍述方式會影響學生的思考，原來連發放題目的方式，一樣有影響力。

筆者最初設計這個課堂活動時，沒有刻意讓學生找出兩條垂直線的角平分線。給予兩條垂直線，只是貪圖方便。當看見學生各種創意解法後，便發現可讓學生討論的空間更為廣濶。當然，要讓學生欣賞一題多解，以及了解新舊課題的關聯，解題後帶領全班討論最為重要。

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附錄一

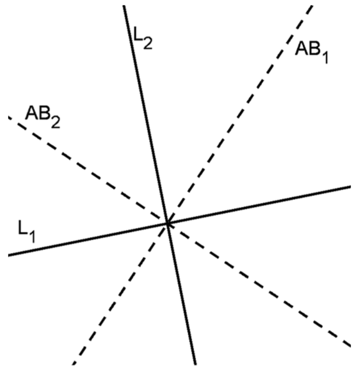
《工作紙三》

The line L_1 with direction $\begin{pmatrix}5\\1\end{pmatrix}$ cuts the y-axis at 2.

The line L_2 passes through (–1 , 7) and is parallel to $\begin{pmatrix}-1\\5\end{pmatrix}$.

Find the equations of the bisectors of the angles between the two given lines.

解：



$$\text{Equation of } L_1\text{: } \mathbf{r} = \begin{pmatrix}0\\2\end{pmatrix} + s\begin{pmatrix}5\\1\end{pmatrix}$$

$$\text{Equation of } L_2\text{: } \mathbf{r} = \begin{pmatrix}-1\\7\end{pmatrix} + t\begin{pmatrix}-1\\5\end{pmatrix}$$

$$\begin{pmatrix}0\\2\end{pmatrix} + s\begin{pmatrix}5\\1\end{pmatrix} = \begin{pmatrix}-1\\7\end{pmatrix} + t\begin{pmatrix}-1\\5\end{pmatrix} \quad \Rightarrow \quad s = 0\ ,\ t = -1$$

The coordinates of point of intersection are (0 , 2) .

$$\text{Direction vector of } AB_1 = \begin{pmatrix}5\\1\end{pmatrix} + \begin{pmatrix}-1\\5\end{pmatrix} = \begin{pmatrix}4\\6\end{pmatrix}$$

$$\therefore \quad \text{equation of } AB_1\text{: } \mathbf{r} = \begin{pmatrix}0\\2\end{pmatrix} + \lambda\begin{pmatrix}4\\6\end{pmatrix}$$

$$\text{Direction vector of } AB_2 = \begin{pmatrix}-5\\-1\end{pmatrix} + \begin{pmatrix}-1\\5\end{pmatrix} = \begin{pmatrix}-6\\4\end{pmatrix}$$

$$\therefore \quad \text{equation of } AB_2\text{: } \mathbf{r} = \begin{pmatrix}0\\2\end{pmatrix} + \mu\begin{pmatrix}-6\\4\end{pmatrix}$$

附錄二

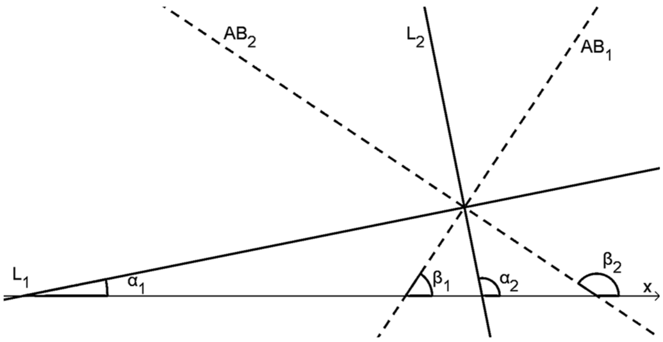
《工作紙二》

The line L_1 with slope $\frac{1}{5}$ passes through (0 , 2).

The line L_2 passes through (–1 , 7) and (1 , –3).

Find the equations of the bisectors of the angles between the two given lines.

解一：考慮角平分線的斜率



Equation of L_1: $y = \frac{1}{5}x + 2$

Equation of L_2: $y - 7 = \frac{-3-7}{1-(-1)}(x+1)$

$$y = -5x + 2$$

L_1 and L_2 intersect at (0 , 2).

$$\tan\alpha_1 = \frac{1}{5} \qquad \tan\alpha_2 = -5$$

$$\frac{2\tan\frac{\alpha_1}{2}}{1-\tan^2\frac{\alpha_1}{2}} = \frac{1}{5} \qquad \frac{2\tan\frac{\alpha_2}{2}}{1-\tan^2\frac{\alpha_2}{2}} = -5$$

$$\tan^2\frac{\alpha_1}{2} + 10\tan\frac{\alpha_1}{2} - 1 = 0 \qquad 5\tan^2\frac{\alpha_2}{2} - 2\tan\frac{\alpha_2}{2} - 5 = 0$$

$$\tan\frac{\alpha_1}{2} = -5 + \sqrt{26} \qquad \tan\frac{\alpha_2}{2} = \frac{1+\sqrt{26}}{5}$$

$$\beta_1 = \alpha_1 + \frac{\alpha_2 - \alpha_1}{2} = \frac{\alpha_1 + \alpha_2}{2}$$

$$\tan\beta_1 = \tan\frac{\alpha_1 + \alpha_2}{2}$$

$$= \frac{\tan\frac{\alpha_1}{2} + \tan\frac{\alpha_2}{2}}{1 - \tan\frac{\alpha_1}{2}\tan\frac{\alpha_1}{2}}$$

$$= \frac{-5+\sqrt{26}+\dfrac{1+\sqrt{26}}{5}}{1-(-5+\sqrt{26})(\dfrac{1+\sqrt{26}}{5})}$$

$$= \frac{3}{2}$$

$$\tan\beta_2 = \tan(\beta_1 + 90°)$$

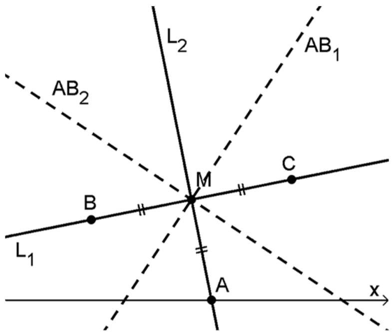
$$= -\frac{1}{\tan\beta_1}$$

$$= -\frac{2}{3}$$

Equation of AB_1: $y = \dfrac{3}{2}x + 2$

Equation of AB_2: $y = -\dfrac{2}{3}x + 2$

解二：考慮與線等距的點的軌跡



Equation of L_1: $y = \dfrac{1}{5}x + 2$

Equation of L_2: $y - 7 = \dfrac{-3-7}{1-(-1)}(x+1)$

$y = -5x + 2$

The coordinates of M and A are (0 , 2) and (0.4 , 0) respectively.

Length of $MA = \sqrt{0.4^2 + 2^2} = \sqrt{4.16}$

$$\begin{cases} y = \frac{1}{5}x + 2 \\ \sqrt{4.16} = \sqrt{x^2 + (y-2)^2} \end{cases}$$

$\therefore$ the coordinates of B and C are (–2 , 1.6) and (2 , 2.4) respectively.

Equation of AB_1:

$$\begin{aligned} \sqrt{(x-0.4)^2 + y^2} &= \sqrt{(x+2)^2 + (y-1.6)^2} \\ x^2 - 0.8x + 0.16 + y^2 &= x^2 + 4x + 4 + y^2 - 3.2y + 2.56 \\ 3.2y &= 4.8x + 6.4 \\ y &= \frac{3}{2}x + 2 \end{aligned}$$

Equation of AB_2:

$$\begin{aligned} \sqrt{(x-0.4)^2 + y^2} &= \sqrt{(x-2)^2 + (y-2.4)^2} \\ x^2 - 0.8x + 0.16 + y^2 &= x^2 - 4x + 4 + y^2 - 4.8y + 5.76 \\ 4.8y &= -3.2x + 9.6 \\ y &= -\frac{2}{3}x + 2 \end{aligned}$$