Some Properties of the Orthocentre of a Triangle

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Introduction

Orthocentre is one of the four special points of a triangle that are included in the KS3 Mathematics Curriculum in Hong Kong. As the point of concurrence of the three altitudes of a triangle, students should have no difficulty in constructing it. However, it seems that not many geometric properties are related to the orthocentre comparing to the other three special points of a triangle. The paper is trying to illustrate some properties of the orthocentre that is not commonly known to our students.

One Figure, Four Orthocentres

For an acute-angled triangle *ABC*, the orthocentre *H* can be easily constructed by joining the three altitudes (Figure 1). Note that *H* lies inside the triangle *ABC*.

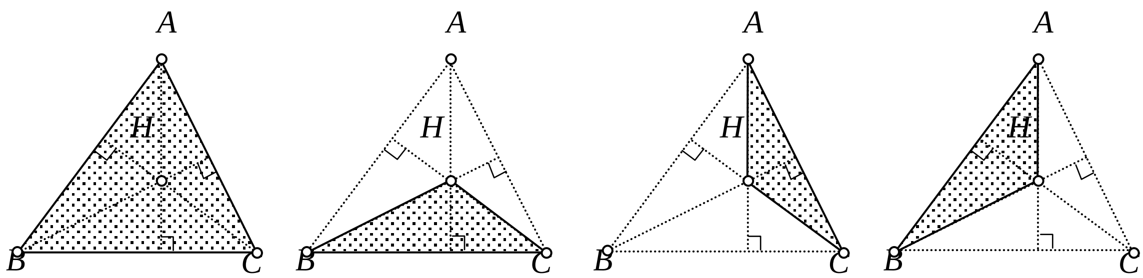

Figure 1 The Fourth Points are the Orthocentres

However if we consider the triangles *HBC*, *HCA* and *HAB*, the fourth points *A*, *B* and *C* become the orthocentres of these triangles respectively (Figure 1). The result follows if we consider the six line segments in the figure as three pairs of perpendicular lines.

These four triangles and their altitudes and orthocentres not only share the same figure, their circumcircles are congruent! (Figure 2)

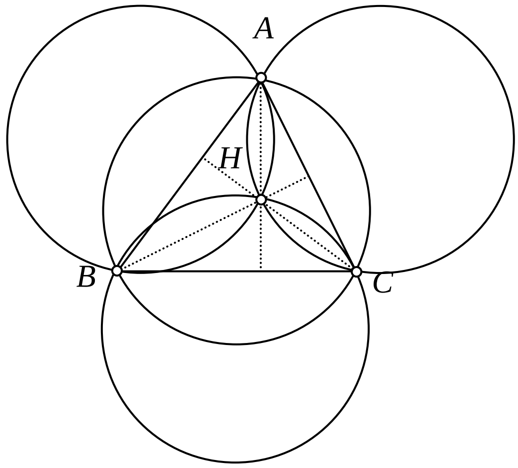

Figure 2

Four Congruent Circles: The four circumcircles of △*ABC*, △*HBC*, △*AHC* and △*ABH* are congruent

In order to prove this, we need the following lemma.

Lemma: If *AH* cuts the side *BC* and the circumcircle of *ABC* at *X* and *Y*, then *HX* = *XY*. (Figure 3)

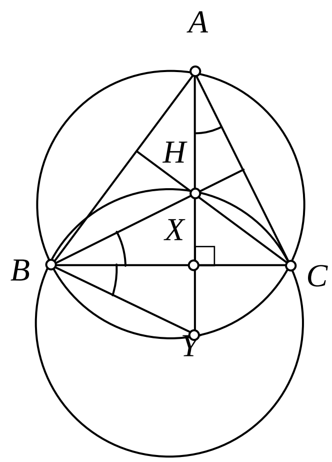

Figure 3

Proof of Lemma: Note that both ∠*HBC* and ∠*HAC* are $90^{\circ} - C$ by the altitudes *HB* and *HA* respectively. But ∠*HAC* = ∠*YAC* = ∠*YBC* as angles in the same segment. Hence ∠*HBX* = ∠*YBX*. Note that ∠*HXB* = ∠*YXB* = 90° and *BX* is common. It follows that △*HBX* and △*YBX* are congruent by SAS. Hence *HX* = *XY*.

The point *Y* can be viewed as the image of *H* by reflection along *BC*. Hence △*HBC* and △*YBC* are congruent and their circumcircles are congruent as well, but the circumcircle of △*YBC* is the circumcircle of △*ABC*.

It can be proved with the same argument that the other two circumcircles are also congruent to the circumcircle of △*ABC*.

Nine Mid-points and One Circle

The circumcircle of △*HBC* is congruent to the circumcircle of △*ABC* as the reflected image along *BC* where *Y* is the reflected image of *H* and *X* is the mid-point of *HY*. However we can also interpret the circumcircle of △*ABC* as the image of the circumcircle of △*HBC* by the **rotation** about the mid-point *A'* of *BC* by 180°. Let *H*' be the image of *H*, which lies on the circumcircle of △*ABC*. It follows that *HA'H'* is a straight line and *HA'* = *A'H'*. (Figure 4)

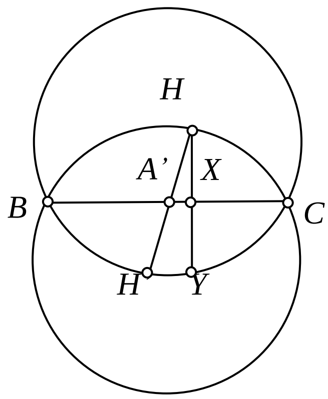

Figure 4

We now have the mid-points *A'* and the foot of altitude *X* on the side *BC*, both of them are mid-points of line segments with one end at *H* and the other end on the circumcircle. If we consider the similar constructions on the other two sides *CA* and *AB* of the triangle, the feet of altitudes and the mid-points are also mid-points of line segments with one end at *H* and the other end on the circumcircle of △*ABC*. Finally if we include the mid-points of *HA*, *HB* and *HC*, we have nine mid-points, all are the mid-points of line segments with one end H and the other end on the circumcircle. (Figure 5)

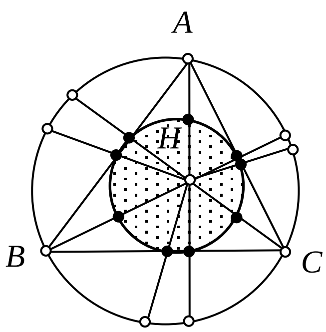

Figure 5

These nine points are co cyclic! The circle is the reduction of the circumcircle centered at the orthocentre H with a scale factor of $\frac{1}{2}$. Note that the nine points are the images of the corresponding points on the circumcircle. The circle passing through these nine points is the well-known Nine-point circle.

References

Co Exeter, H.S.M & Greater S.L. (1967). *Geometry Revisited.* Mathematical Association of America

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