Solving Cubic Equations by Viète's Substitutions

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Introduction

In his article in *EduMath* 27, Leung Chi Kit illustrates how to solve a polynomial equation by the trigonometric substitution $x = \cos\theta$ and by solving the obtained equation $\cos 6\theta = 6$ using Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ (Leung, 2008). This very interesting method involves Euler's formula which is beyond the matriculation level. This article discusses alternative ways to solve the polynomial equations in Leung's article which only requires mathematics knowledge within matriculation level.

These alternative ways are the ingenious substitutions used by French mathematician François Viète (1540 – 1603) to solve cubic equations (Hollingdale, 1989, p.122). Simple examples would first be used to illustrate how these substitutions work. Afterward the second equation in Leung's article would be solved using Viète's substitutions.

Viète's substitutions

Given a general cubic equation $x^3 + ax^2 + bx + c = 0$, it is well known that the substitution $x = y - \dfrac{a}{3}$ could transform the equation into the form $y^3 + py + q = 0$. Viète's mastery of trigonometry enabled him to solve this equation using the triple angle formula $\cos 3A = 4\cos^3 A - 3\cos A$. He first puts $y = k\cos A$ into the equation $y^3 + py + q = 0$, which gives

$$k^3\cos^3 A + pk\cos A + q = 0.$$

Suppose k could be chosen in the way such that $k^3 : pk = 4 : -3$, that is,

$$\frac{k^3}{pk} = -\frac{4}{3}$$

$$\Rightarrow \qquad k^2 = -\frac{4p}{3}.$$

If $p < 0$, choose $k = \sqrt{-\frac{4p}{3}}$ and put $y = \sqrt{-\frac{4p}{3}} \cos A$. So, we have

$$-\frac{4p}{3}\sqrt{-\frac{4p}{3}}\cos^3 A + p\sqrt{-\frac{4p}{3}}\cos A + q = 0$$

$$\Rightarrow \quad (4\cos^3 A - 3\cos A)\left(-\frac{p}{3}\sqrt{-\frac{4p}{3}}\right) = -q$$

$$\Rightarrow \quad \cos 3A = \frac{q}{\frac{p}{3}\sqrt{-\frac{4p}{3}}}$$

Hence the cubic equation is reduced to a simple trigonometric equation. For this equation to have real solutions,

$$\left|\frac{q}{\frac{p}{3}\sqrt{-\frac{4p}{3}}}\right| \le 1$$

$$\Leftrightarrow \quad \frac{q^2}{\frac{p^2}{9}\times\frac{-4p}{3}} \le 1 \quad (\because\ p < 0)$$

$$\Leftrightarrow \quad \frac{p^3}{27} + \frac{q^2}{4} \le 0 \quad (\because\ p < 0)$$

Hence if $\frac{p^3}{27} + \frac{q^2}{4} \le 0$, we could solve the equation $y^3 + py + q = 0$ by the substitution $y = \sqrt{-\frac{4p}{3}} \cos A$ and the triple angle formula.

EXAMPLE 1 Solve $x^3 - 12x + 8 = 0$.

SOLUTION Put $x = \sqrt{-\frac{4(-12)}{3}} \cos A = 4\cos A$. The equation becomes

$$64\cos^3 A - 48\cos A + 8 = 0$$

$$\Rightarrow \quad 4\cos^3 A - 3\cos A = -\frac{1}{2}$$

$$\Rightarrow \quad \cos 3A = -\frac{1}{2}$$

$$\Rightarrow \qquad A = 120°\cdot n \pm 40°, \quad \text{where} \quad n \in \mathbf{Z}$$

Hence, we have

$$\begin{aligned} x &= 4\cos A \\ &= 4\cos 40°, \quad 4\cos 80° \quad \text{or} \quad 4\cos 160° \\ &= 3.064, \quad 0.6946 \quad \text{or} \quad -3.759 \quad \text{(to 4 sig. fig.)} \end{aligned}$$

If $\frac{p^3}{27}+\frac{q^2}{4}>0$, Viète puts $y=z-\frac{p}{3z}$ in the equation $y^3 + py + q = 0$ to give

$$\left(z-\frac{p}{3z}\right)^3+p\left(z-\frac{p}{3z}\right)+q = 0$$

$$\Rightarrow \qquad z^3-\frac{p^3}{27z^3}+q = 0$$

This quadratic equation in z^3 is then solved to give

$$z^3 = -\frac{q}{2}\pm\sqrt{\frac{p^3}{27}+\frac{q^2}{4}} \quad .$$

This equation gives only one real z . Hence the original equation also has only one real root.

EXAMPLE 2 Solve $x^3 - 12x + 20 = 0$.

SOLUTION Note that only the constant term of this equation is different from that in EXAMPLE 1. If we put $x = 4\cos A$ again, the equation becomes

$$\cos 3A = -\frac{5}{4} \quad , \quad \text{which has no real solution.}$$

In this case, putting $x=z-\frac{(-12)}{3z}=z+\frac{4}{z}$, we have

$$\left(z+\frac{4}{z}\right)^3-12\left(z+\frac{4}{z}\right)+20 = 0$$

$$z^3+\frac{64}{z^3}+20 = 0$$

$$z^3 = -4 \quad \text{or} \quad -16$$

$$z = -2^{\frac{2}{3}}, \ -2^{\frac{2}{3}}\,\mathrm{cis}(\pm 120°) \quad \text{or} \quad -2^{\frac{4}{3}}, \ -2^{\frac{4}{3}}\,\mathrm{cis}(\pm 120°),$$

where $\mathrm{cis}(\pm 120°) = \cos 120° \pm i \sin 120°$

$$\Rightarrow \quad x = -2^{\frac{2}{3}} - 2^{\frac{4}{3}},\ -2^{\frac{2}{3}}\text{cis}(\pm 120°) \text{ or } -2^{\frac{4}{3}}\text{cis}(\mp 120°)$$

Note that the two sets of values of z give only one set of values of x.

Solving the equation by Viète's substitutions

Let us now look at the second equation in Leung's article: $64x^6 - 96x^4 + 36x^2 - 8 = 0$. It is clear that this equation is a cubic equation in x^2. Put $x^2 = \frac{y+1}{2}$, we have

$$64\left(\frac{y+1}{2}\right)^3 - 96\left(\frac{y+1}{2}\right)^2 + 36\left(\frac{y+1}{2}\right) - 8 = 0$$

$$\Rightarrow \quad 4y^3 - 3y - 3 = 0$$

Obviously Viète's first substitution $y = \cos A$ gives the equation $\cos 3A = 3$ which has no real solutions of A. Let's use Viète's second substitution $y = z + \frac{1}{4z}$. We have

$$4\left(z + \frac{1}{4z}\right)^3 - 3\left(z + \frac{1}{4z}\right) - 3 = 0$$

$$\Rightarrow \quad 4z^3 + \frac{1}{16z^3} - 3 = 0$$

$$\Rightarrow \quad z^3 = \frac{3 \pm 2\sqrt{2}}{8}$$

Let $\alpha = \frac{3+2\sqrt{2}}{8}$ and $\beta = \frac{3-2\sqrt{2}}{8}$ (note that $\alpha\beta = \frac{1}{64}$) and we have

$$z = \alpha^{\frac{1}{3}},\ \alpha^{\frac{1}{3}}\text{cis}(\pm 120°) \text{ or } \beta^{\frac{1}{3}},\ \beta^{\frac{1}{3}}\text{cis}(\pm 120°).$$

Hence $y = z + \frac{1}{4z}$

$$\Rightarrow \quad = \alpha^{\frac{1}{3}} + \beta^{\frac{1}{3}} \text{ or } \alpha^{\frac{1}{3}}\text{cis}(\pm 120°) + \beta^{\frac{1}{3}}\text{cis}(\mp 120°) \text{ as } \beta^{\frac{1}{3}} = \frac{1}{4\alpha^{\frac{1}{3}}}.$$

Finally let us first consider the real roots of x. Since $x = \pm\sqrt{\frac{y+1}{2}}$, we have

$$x = \pm\sqrt{\frac{\alpha^{\frac{1}{3}} + \beta^{\frac{1}{3}} + 1}{2}}$$

$$= \pm\sqrt{\frac{(\alpha^{\frac{1}{6}} + \beta^{\frac{1}{6}})^2 - 2\alpha^{\frac{1}{6}}\beta^{\frac{1}{6}} + 1}{2}}$$

$$= \pm\sqrt{\frac{(\alpha^{\frac{1}{6}} + \beta^{\frac{1}{6}})^2 - 2\times\frac{1}{2} + 1}{2}} \quad \text{since} \quad \alpha\beta = \frac{1}{64}$$

$$= \pm\frac{\alpha^{\frac{1}{6}} + \beta^{\frac{1}{6}}}{\sqrt{2}}$$

$$= \pm\frac{1}{2}((3+\sqrt{2})^{\frac{1}{6}} + (3-\sqrt{2})^{\frac{1}{6}})$$

For the complex roots of x,

$$x = \pm\sqrt{\frac{\alpha^{\frac{1}{3}}\text{cis}(\pm120°) + \beta^{\frac{1}{3}}\text{cis}(\mp120°) + 1}{2}}$$

$$= \pm\sqrt{\frac{(\alpha^{\frac{1}{6}}\text{cis}(\pm60°) + \beta^{\frac{1}{6}}\text{cis}(\mp60°))^2}{2}} \quad \text{since} \quad \alpha\beta = \frac{1}{64}$$

$$= \pm\frac{\alpha^{\frac{1}{6}}\text{cis}(\pm60°) + \beta^{\frac{1}{6}}\text{cis}(\mp60°)}{\sqrt{2}}$$

$$= \pm\frac{1}{2}((3+2\sqrt{2})^{\frac{1}{6}}\text{cis}(\pm60°) + (3-2\sqrt{2})^{\frac{1}{6}}\text{cis}(\mp60°))$$

Readers can check that these six roots are identical to those in Leung's article. Since $3 \pm 2\sqrt{2} = (\sqrt{2} \pm 1)^2$, these six roots can be further simplified as $\pm\frac{1}{2}((\sqrt{2}+1)^{\frac{1}{3}} + (\sqrt{2}-1)^{\frac{1}{3}})$ and $\pm\frac{1}{2}((\sqrt{2}+1)^{\frac{1}{3}}\text{cis}(\pm60°) + (\sqrt{2}-1)^{\frac{1}{3}}\text{cis}(\mp60°))$.

Final Remarks: Cardano's *Casus Irreducibilis*

Gerolamo Cardano (1501–1576) gives a formula for solving cubic equations, now known as Cardano's formula, which states that if $x^3 + px + q = 0$, $x = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}}$. He was deeply puzzled when he applied the formula to the equation $x^3 - 15x - 4 = 0$ and obtained $x = \sqrt[3]{2+\sqrt{-121}} + \sqrt[3]{2-\sqrt{-121}}$ instead of the simple solution

$x = 4$. He called this situation a *casus irreducibilis* (Hollingdale, 1989 p.118). Let us try to solve this equation using Viète's substitutions.

To avoid complex numbers, we have to use Viète's first substitution $x = 2\sqrt{5}\cos A$ and get $40\sqrt{5}\cos^3 A - 30\sqrt{5}\cos A - 4 = 0$

$$\Rightarrow \qquad \cos 3A = \frac{2}{5\sqrt{5}}$$

We can use calculators to find the values of $3A$, and then of A and finally of x, but then we are not able to tell the exact values of the roots. Alternatively, since $x = 4$ and $x = 2\sqrt{5}\cos A$ we can tell that $\cos A = \frac{2}{\sqrt{5}}$ (readers can check that $\cos 3A = \frac{2}{5\sqrt{5}}$ indeed), and we can find the exact values of the other two roots from $x = 2\sqrt{5}\cos(A \pm 120°)$. However, this is unnecessarily tedious as we can just solve the equation by factorizing it as $(x - 4)(x^2 + 4x + 1) = 0$ in the first place. Therefore Viète's substitution doesn't help much in this case.

The moral of the story is that we should not take a musket to kill a butterfly （殺雞焉用牛刀）. If a cubic equation has rational roots, we better solve it using factor theorem instead of using Viète's substitutions or Cardano's formula. Otherwise some oddities could come up, although these oddities could also provide us some interesting exercises on complex numbers and trigonometry, such as showing that $\sqrt[3]{2+\sqrt{-121}} + \sqrt[3]{2-\sqrt{-121}} = 4$.

References

Hollingdale, S. (1989). *Makers of Mathematics.* London: Penguin Books.

Leung, C.K. (2008). Solving Special Polynomial Equations by Trigonometric Substitution. *EduMath 27*, p.88.

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