Solving Special Polynomial Equations by Trigonometric Substitution

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Let's study the following question.

> *Show that* $\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$.
> *Hence, solve the equation* $64x^6 - 96x^4 + 36x^2 - 3 = 0$.

The first half of this question can be shown by comparing the real parts after expanding the right hand side of $\cos 6\theta + i\sin 6\theta = (\cos\theta + i\sin\theta)^6$. For the second half of the question, we substitute x by $\cos\theta$ to get

$$
\begin{aligned}
& 64\cos^6\theta - 96\cos^4\theta + 36\cos^2\theta - 3 = 0 \\
\Rightarrow\quad & 64\cos^6\theta - 96\cos^4\theta + 36\cos^2\theta - 2 = 1 \\
\Rightarrow\quad & \cos 6\theta = \frac{1}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (*) \\
\Rightarrow\quad & \theta = \frac{k\pi}{3} \pm \frac{\pi}{18}, \quad \text{where } k \in \mathbf{Z}
\end{aligned}
$$

Therefore, the roots of the above equation are $\pm\cos\frac{\pi}{18}$, $\pm\cos\frac{5\pi}{18}$ and $\pm\cos\frac{7\pi}{18}$. These roots can be illustrated by plotting the graph of $y = 64x^6 - 96x^4 + 36x^2 - 3$ as in Figure 1.

The above question is a typical example in solving special polynomial equations using trigonometric substitution. I have used it in my pure mathematics classes for many years. However, after I had demonstrated the above example to my class this year, one of my students was not satisfied. He wondered why we could assume the absolute values of the roots being all less than or equal to 1 so that we could substitute x by $\cos\theta$ *before* we solved the equation. Furthermore, what would happen if the constant of the right hand side of equation (*) was greater than 1 ? In particular, if we were asked to solve the equation $64x^6 - 96x^4 + 36x^2 - 8 = 0$, we would get $\cos 6\theta = 3$ after letting $x = \cos\theta$. We could not claim the equation to be not solvable since the graph of the corresponding polynomial passed through the x-axis as shown in Figure 2.

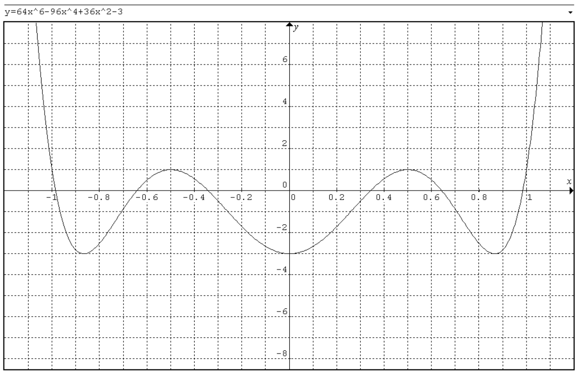

Figure 1

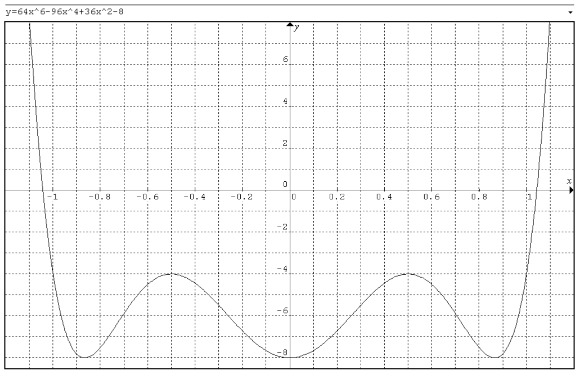

Figure 2

After analyzing the problem for a while, I find out that the above method of trigonometric substitution does work for solving equations like $64x^6 - 96x^4 + 36x^2 - 8 = 0$ if we extend the domain of definition of the cosine function to the field of complex numbers.

In fact, from the Euler's Formula, $e^{i\phi} = \cos\phi + i\sin\phi$, we have

$$\cos\phi = \frac{e^{i\phi} + e^{-i\phi}}{2} \quad .$$

Replacing the real value ϕ by a complex number $a + bi$, we have

$$\begin{aligned}\cos(a+bi) &= \frac{e^{i(a+bi)} + e^{-i(a+bi)}}{2} = \frac{e^{-b+ai} + e^{b-ai}}{2} \\ &= \frac{1}{2}[e^{-b}(\cos a + i\sin a) + e^{b}(\cos a - i\sin a)] \\ &= \frac{1}{2}[(e^{-b} + e^{b})\cos a + i(e^{-b} - e^{b})\sin a] .\end{aligned}$$

If $\cos(a + bi) = 3$, it is equivalent to solve the system of equations

$$\begin{cases} (e^{-b} + e^{b})\cos a = 6 & \text{.......................................(1)} \\ (e^{-b} - e^{b})\sin a = 0 & \text{.......................................(2)} \end{cases}$$

Since b must be non-zero, we have $a = 0$ from equation (2). Then, equation (1) becomes $e^{-b} + e^{b} = 6$ or $e^{2b} - 6\,e^{b} + 1 = 0$. By quadratic formula, we have $e^{b} = 3 \pm 2\sqrt{2}$ or $b = \ln(3 \pm 2\sqrt{2})$. That is to say, $\cos(i\ln(3 + 2\sqrt{2})) = 3$ (I will explain why I choose the positive sign soon).

As we are solving $\cos 6\theta = 3$, we may put $6\theta = i\ln(3 + 2\sqrt{2})$ and this implies $\theta = i\ln(3+2\sqrt{2})^{\frac{1}{6}}$.

$$\text{Finally,}\quad x = \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{e^{-\ln(3+2\sqrt{2})^{\frac{1}{6}}}+e^{\ln(3+2\sqrt{2})^{\frac{1}{6}}}}{2}$$

$$= \frac{1}{2}\left[\frac{1}{(3+2\sqrt{2})^{\frac{1}{6}}}+(3+2\sqrt{2})^{\frac{1}{6}}\right] .$$

It can be checked by calculator that the above value is indeed a root of the equation $64x^6 - 96x^4 + 36x^2 - 8 = 0$!

I would like to point out that the above solution is by no means perfect. For instance, only a positive root is obtained. In fact, when we are taking the sixth root of $3 + 2\sqrt{2}$, it is equivalent to solve the equation $z^6 = 3 + 2\sqrt{2}$. So, we should put $\theta_k = i\ln[(3+2\sqrt{2})^{\frac{1}{6}}\,\omega^k]$ where $\omega = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$ and $k = 0, 1, 2, \ldots, 5$. In this way, six (complex) roots of x are obtained by putting $x = \cos\theta_k$ where $k = 0, 1, 2, \ldots, 5$. Detailed computation is left to readers as exercise.

Besides, the same result will be obtained if $3 + 2\sqrt{2}$ is replaced by $3 - 2\sqrt{2}$ in the above solution since $\frac{1}{3-2\sqrt{2}} = 3 + 2\sqrt{2}$. So, putting $3 = \cos(i\ln(3 + 2\sqrt{2}))$ is the same as putting $3 = \cos(i\ln(3 - 2\sqrt{2}))$.

As a conclusion, the method of trigonometric substitution does work for different values for the constant term of a polynomial equation although it requires a mathematical concept that goes beyond the matriculation level.

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