On Euler's formula

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When teaching the general solution of the linear, homogeneous, second-order differential equation

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0 \quad \text{..............(1)}$$

where a, b, c are constants with $b^2 - 4ac < 0$, many teachers face a problem of using Euler's formula

$$e^{i\theta}=\cos\theta+i\sin\theta$$

to simplify the general solution from $y=Ae^{(p+iq)x}+Be^{(p-iq)x}$ to $y=e^{px}(C\cos qx+D\sin qx)$, where $p\pm qi$ are the complex roots of the auxiliary equation $a\lambda^2 + b\lambda + c = 0$. The difficulty is that Euler's formula involves complex variables and is usually proved using Taylor's series which is unfamiliar to sixth-form students. The following suggests a simple way of establishing the validity of Euler's formula.

Let $y = \cos\theta + i\sin\theta$. Then

$$\begin{aligned}\frac{dy}{d\theta}&=\frac{d}{d\theta}(\cos\theta+i\sin\theta)\\&=\frac{d}{d\theta}(\cos\theta)+i\frac{d}{d\theta}(\sin\theta) && \text{since } i \text{ is a constant}\\&=-\sin\theta+i\cos\theta\\&=i(\cos\theta+i\sin\theta) && \text{since } i^2=-1\\&=iy\end{aligned}$$

Therefore y satisfies a simple first order differential equation

$$\frac{dy}{d\theta}=iy$$

$$\therefore \frac{dy}{y}=i\,d\theta$$

$$\int\frac{dy}{y}=i\int d\theta$$

$$\therefore \ln y=i\theta+C \qquad \text{where } C \text{ is an arbitrary constant}$$

$$\text{i.e. } y=C\,e^{i\theta} \qquad \text{where } C=e^C$$

Since when $\theta = 0$, $y = \cos 0 + i \sin 0 = 1$, we have $C = 1$.

$$\therefore y = e^{i\theta}$$

i.e. $\cos\theta + i\sin\theta = e^{i\theta}$

As stated by Mr. Cheung Pak Hong in [1], students may be puzzled by the complex variables involved in Euler's formula. To help the students understand Euler's formula better, teachers can remind the students that the function $f(\theta) = \cos\theta + i\sin\theta$ satisfies the special property that for all θ and ϕ,

$$\begin{aligned} f(\theta)f(\phi) &= (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) \\ &= (\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\sin\theta\cos\phi + \cos\theta\sin\phi) \\ &= \cos(\theta + \phi) + i\sin(\theta + \phi) \\ &= f(\theta + \phi) \end{aligned}$$

Note that the exponential function $f(x) = a^x$ (a is a constant) satisfies this property, and it is therefore sensible to say that $f(\theta) = \cos\theta + i\sin\theta$ is an exponential function. In fact it can be proved that if a function f is non-zero, differentiable and $f(x + y) = f(x)f(y)$ for all x and y, then $f(x) = e^{\alpha x}$ where $\alpha = f'(0)$ (See [2] or [3]).

Note that if teachers and students are still uncomfortable of using complex numbers to find the real solutions of a real differential equation, they could be satisfied by a method suggested by Mr. Cheung Pak Hong in [1]. This method has the advantage of not using complex numbers, but at the price of losing the convenience and efficiency of solving (1) by auxiliary equations.

Reference

1. Cheung, P.H. Teaching Differential Equations in School: Can Complex Numbers Be Abandoned? *Teaching Mathematics and its Applications*, 1993, 12(1), pp.32-33.
2. Hardy, G.H. *A Course of Pure Mathematics*, (ELBS), (1944), pp.408.
3. Spivak, M. *Calculus*, (Addison Wesley), (1967), pp.300.