A Discourse of a Mathematics Lesson about 136 and 316

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1. Introduction

Some years ago, I presented the following problem as a worked example in an algebra lesson for my S1 students. It is a very popular and funny problem in which I would like to share my teaching strategies with the readers. It also serves as a record for my teaching life so that I could possibly refine my teaching approaches in the near future. Let's begin the whole story with the following question.

***Question: In a farm, there are hens and pigs. If there are altogether 136 heads and 316 legs, how many pigs are there in the farm?**

After viewing the above problem, one student suggested Method 1 quickly:

- (Method 1) Let x be the number of pigs. Then number of hens = $136 - x$.
  Hence, we have $4x + 2(136 - x) = 316$.
  $4x + 272 - 2x = 316$
  $2x = 44$
  $x = 22$
  So, there are 22 pigs in the farm.

Teacher praised the student who made such a first attempt without any error. After a short while, another student was eager to share another idea, that is Method 2:

- (Method 2) Let x be the number of pigs. Then number of hens = $136 - x$.
  Hence, we have $5x + 3(136 - x) = 316 + 136$.
  $5x + 408 - 3x = 452$

$2x = 44$
$x = 22$ and so the result follows.

After the presentation of this solution, a few students asked for the meaning of the L.H.S. of the equation. Then, the student told the whole class the following:

(1) $5x$ refers to the total number of heads and legs (i.e. 1 head + 4 legs for each pig) for all pigs in the farm;

(2) $3(136 - x)$ refers to the total number of heads and legs (i.e. 1 head + 2 legs for each hen) for all hens in the farm.

Teacher clapped his hands for the good presentation of a new idea like this. He also supplemented student's findings by pointing out that this solution is a bit more complicated than Method 1 and it gives no extra benefit at all. In particular, it can also be regarded as merging the following two equations by addition:

$$\begin{cases} 4x + 2(136 - x) = 316 & ----------(1) \\ x + (136 - x) = 136 & -----------(2) \end{cases}$$

Next, teacher asked the whole class, "Are there other possible solutions? Say, how about the use of two variables?"

To the teacher's surprise, another student put up his hand and then suggested Method 3:

- (Method 3 – Use 2 variables) Let x and y be the number of pigs and hens respectively. Then we have
  $x + y = 136 ----(1)$
  $4x + 2y = 316 ----(2)$
  $(1) \times 4$: $4x + 4y = 544 ----(3)$
  $(3) - (2)$: $2y = 228 \Rightarrow y = 114$
  i.e. $x = 136 - 114 = 22$
  So, there are 22 pigs in the farm.

As solving simultaneous equations is a topic in S2 instead of S1, teacher felt very impressive to notice that the student was able to solve it himself. When teacher invited for more comments from other students in class, many students claimed that this method is easier to understand and they solved it in a similar manner too.

At this moment, the teacher took the opportunity to prompt students to further refine the above solutions. More precisely, the teacher asked, "Could we solve the problem without using any variables at all?"

It was an amazing incident that another student voiced out the following method soon afterwards:

- (Method 4 – without using equation, assume all are pigs and then do adjustment)
  Number of hens = $\frac{136\times4-316}{4-2} = 114.$
  So, number of pigs = $136 - 114 = 22.$
  Here, the student also remarked that the key to solve the problem is just due to the fact that each pig has 2 more legs than that of each hen.

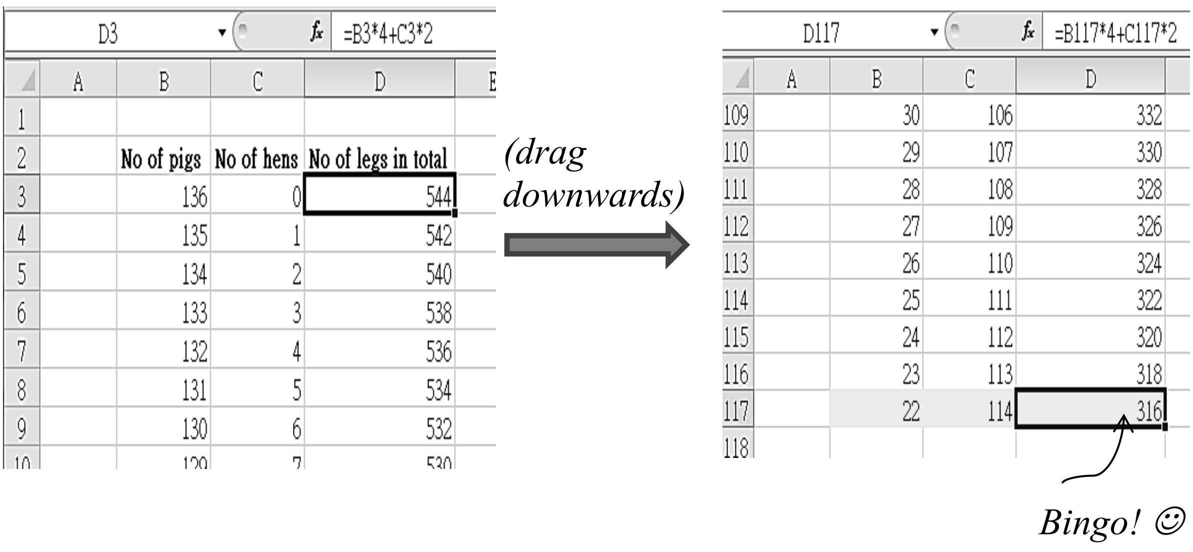
Teacher asked the student why she could come up with such an elegant and funny solution so quickly. Her reply was that she has learnt something similar in primary school before especially during the Mathematics Olympiad training tutorials. Certainly, the teacher grabbed this golden chance by prompting students to think about what would happen if we tried to assume all animals in the farm are hens instead. As a result, another student shared the following solution presented in Method 5:

- (Method 5 – without using equation, assume all are hens and then do adjustment)
  Number of pigs = $\frac{316-136\times2}{4-2} = 22.$
  So, number of hens = $136 - 22 = 114.$

After checking the details of the above solution, the teacher tried to remind students that Method 4 and 5 are very similar except that the roles are being interchanged only. But then some students still found it a bit difficult to understand. As a result, one student has written a lot of draft works in her paper and then suggested the following strategy for deducing the results in a step by step manner. (See Method 6).

- (Method 6 – without using equation and do adjustment by replacing one pig by a hen every time)
  136 pigs $\Longrightarrow$ $136 \times 4 = 544$ legs
  135 pigs + 1 hen $\Longrightarrow$ 542 legs
  $\vdots$
  22 pigs $+ 114$ hens $\Longrightarrow$ 316 legs
  Here, we notice that $\frac{544-316}{2} = 114$ and so there must be 114 hens and 22 pigs.
  The result follows.

The teacher felt extraordinary happy to see that the student was able to do the analysis step by step though she described her idea in a rather "loose" manner. Thus, the teacher tried to login his computer and used a spreadsheet (e.g. MS Excel) to help students visualize the key changes and explore the whole process governed by Method 6 more easily. (See the following for instance.)



	A	B	C	D
1				
2		No of pigs	No of hens	No of legs in total
3		136	0	544
4		135	1	542
5		134	2	540
6		133	3	538
7		132	4	536
8		131	5	534
9		130	6	532

	A	B	C	D
109		30	106	332
110		29	107	330
111		28	108	328
112		27	109	326
113		26	110	324
114		25	111	322
115		24	112	320
116		23	113	318
117		22	114	316
118				

As can be seen in the above, Method 6 is simply an "trial and error" approach by setting up an appropriate formula in the spreadsheet (See [2] for instance) which can be updated automatically when the teacher dragged the cursor downwards until reaching the required goal. The advantage is that it is easier to understand compared with Method 4 or Method 5.

After the discussion about the various solutions (Method 1 – 6) presented by students, the teacher also shared his another solution which became the Method 7 in the whole lesson:

- (Method 7 – Supplemented by teacher for further reference)
  First, we try to cut half of the legs for each hen and each pig. Then, we have total number of remaining legs $\frac{316}{2} = 158$.
  Now, note that all hens (become single-leg hen now) have the same number of head and leg. And, for every existence of a pig (become two-leg pig now), there will be an additional one leg incurred compared with the total number of heads.
  Thus, required number of pigs = $158 - 136 = 22$.
  In other words, the number of hens = $136 - 22 = 114$.
  *Remarks: Students generally appreciated this solution very much.

2. Extension problem (Slight modification of the original problem)

In order to aid students understand better for the last Method 7 introduced by the teacher, the teacher intentionally designed a check-question to aid students rehearse what they have learnt previously. Here comes the check-question:

*Check-question: In a shop, there are **2-wheel bicycles** and **3-wheel bicycles**. If there are altogether 28 seats and 76 wheels, how many 2-wheel bicycles are there in the shop? Explain your answer using the idea introduced in Method 6. (Note: You may assume that all bicycles have only one seat.)

*Solution: First, we try to cut half of the wheels for each 2-wheel bicycle and each 3-wheel bicycle. Then, we have: Total number of remaining wheels for two types of vehicles $= \frac{76}{2} = 38$.

Now, note that all 2-wheel bicycles will have the same number of seats and wheels after we take away half number of the wheels from them. And, for every existence of 3-wheel bicycle (i.e. 1.5-wheel bicycle after we take away half number of wheels from it), there will be an additional 0.5 wheels incurred compared with the total number of seats.

Thus, required number of 3-wheel bicycles $= \frac{38-28}{0.5} = 20$.

In other words, the number of 2-wheel bicycles $= 28 - 20 = 8$.

3. Concluding Remarks

The mathematics problem presented here is actually modified from the famous "Chicken-Rabbit in a Cage" problem. (See [1] for instance) As can be seen from ideas presented in Method 6, the use of spreadsheet as a problem-solving strategy could be regarded as a way to add some STEM components into the lesson. As for Method 7, we can also consider teaching our students to write a Python program (a programming language introduced in the new ICT curriculum) to do the implementation. The Python codes and its running results are attached for readers' reference as below:

- Python Code

```
p=int(input("Enter the total number of heads:"))
q=int(input("Enter the total number of legs:"))
x=q/2-p    # number of pigs
y=p-x      # number of hens
print("The total number of pigs is:",int(x))
print("The total number of hens is:",int(y))
```

- Running results using Google Colab (https://colab.research.google.com/):

```
Enter the total number of heads:136
Enter the total number of legs:316
The total number of pigs is: 22
The total number of hens is: 114
```

As a final remark, let p and q be the total number of heads and the total number of legs respectively. Also, let x and y be the number of pigs and the number of hens respectively. The implementation for coding the Python programs can also be done via matrix computation (suitable for Senior Form students taking M2):

From the idea of method 2 in solving simultaneous equations, we have

$$\begin{pmatrix}1 & 1\\ 4 & 2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}p\\ q\end{pmatrix}\Rightarrow\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}1 & 1\\ 4 & 2\end{pmatrix}^{-1}\begin{pmatrix}p\\ q\end{pmatrix}=\begin{pmatrix}-1 & \frac{1}{2}\\ 2 & -\frac{1}{2}\end{pmatrix}\begin{pmatrix}p\\ q\end{pmatrix}$$

Consequently, the corresponding Python codes can be written as follows:

```
import numpy as np
p=int(input("Enter the total number of heads:"))
q=int(input("Enter the total number of legs:"))
M=[[1,1],[4,2]]
N=np.linalg.inv(M)   #inverse matrix
print("The inverse matrix of M is:",N)
s=[p,q]
t=np.dot(N,s)
print("Number of pigs = x =",int(t[0]))
print("Number of hens = y =",int(t[1]))
```

```
Enter the total number of heads:136
Enter the total number of legs:316
The inverse matrix of M is: [[-1.   0.5]
 [ 2.  -0.5]]
Number of pigs = x = 22
Number of hens = y = 114
```

*Note: The library ***numpy*** has to be used here. (See [3],[4] and [5] for instance)

To this end, the author highly recommends the readers to try writing teaching diaries regularly. It helps teachers to prepare for many possible responses (especially those unexpected ones) which will be delivered by students. The teachers can then design the flow of the lesson in a more smooth and coherent manner when they have to teach the same topic next time. Last but not least, teaching scenes recorded can serve as teaching memories for all of us which are kind of positive energy to educators, I deeply believe!

4. Acknowledgement

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5. References

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