Revisiting the Order and Interpretation of Transformations

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Introduction

Last term, after teaching transformations of a function $f(x)$ based on graphical perspectives, a student asked me whether it is possible to determine the order of transformations by analytical method. The question prompted me to re-consider the topic from a different perspective.

According to the *Mathematics Curriculum and Assessment Guide (Secondary 4 - 6)*, students are expected to "understand the transformations of the function $f(x)$ including $f(x)+k$, $f(x+k)$, $kf(x)$ and $f(kx)$ from tabular, symbolic and graphical perspectives." (2007, p.27)

In this exposition, I present three questions on transformations to motivate discussion on ways to determine the order of transformations. In particular, I will discuss this topic from symbolic perspective using examples presented in tabular form.

Three Motivating Questions

Three questions are presented in this section. They were modified from questions I used in my class tests.

Q.1 **Figure 1** shows the graph of $y = f(x)$ and its transformed images $y = g(x)$ and $y = h(x)$.

Which of the following equations **best** represents the graph of $g(x)$?

(a) $g(x) = f(x+4)+2$
(b) $g(x) = f(x-4)+2$

(c) $g(x) = f(x-2)+4$
(d) $g(x) = -f(x+2)+4$

Q.2 The graph of $y = h(x)$ in **Figure 1** can be obtained from the graph of $f(x)$ by four successive transformations. Describe these transformations **fully**.

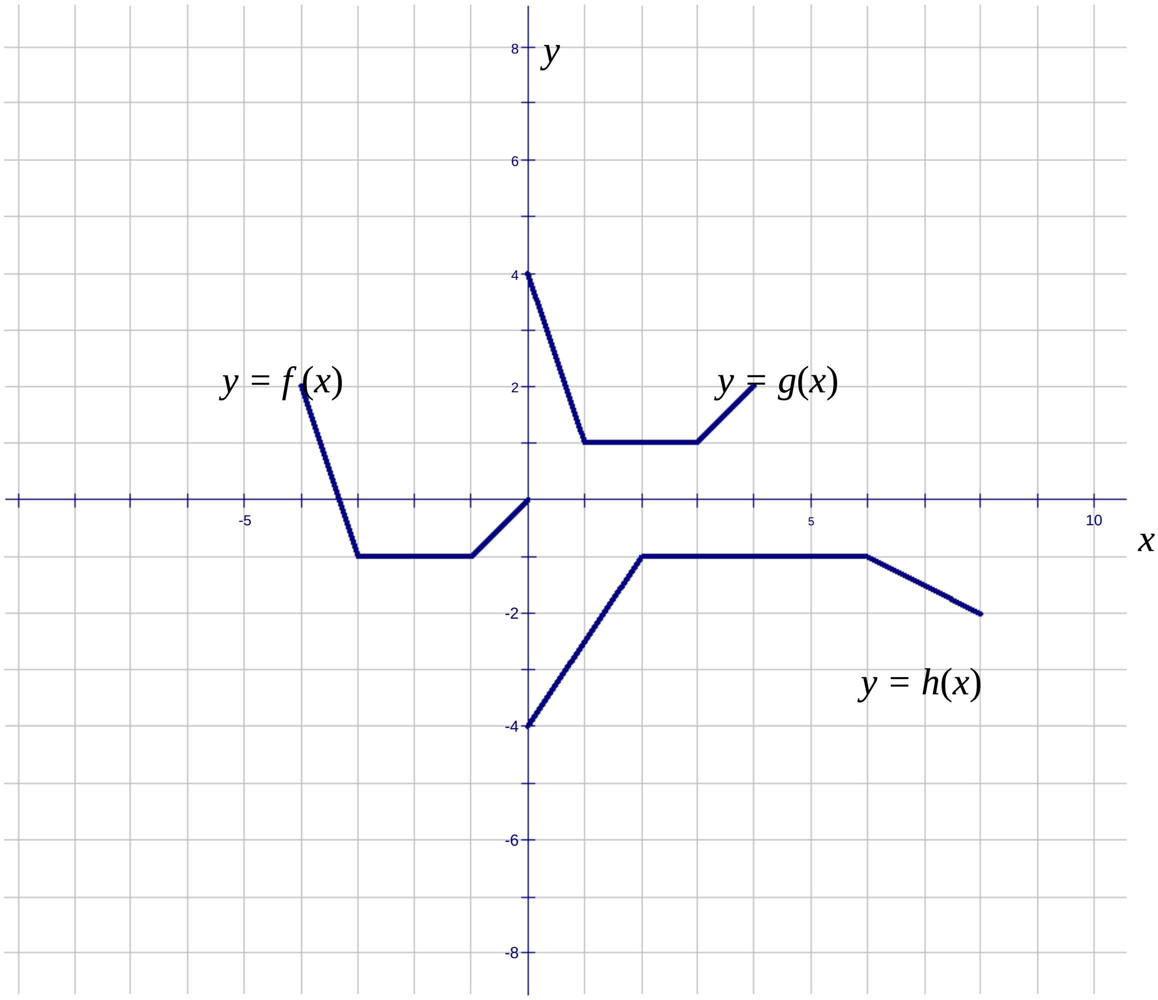

Figure 1 Graphs of $f(x)$, $g(x)$ and $h(x)$

Q3. The graph of $y = m(x)$ is transformed into the graph of $y = m\left(\left|1 - \frac{x}{3}\right|\right) - 5$ by four successive transformations. Describe these transformations **fully**.

Before discussing the solutions of these questions, we will investigate the order and interpretation of transformations.

Note 1: Transformations that only act on the Image

In this section, we will consider transformed functions of the form $y = af(x) + d$, where both the vertical stretch by factor of a and vertical translation by d units act only on the image of the original function $f(x)$. The domain of the transformed function is the same as that of $f(x)$.

Considering the order of operations, the vertical stretch is done before vertical translation. On the other hand, $y = af(x) + d$ could also be expressed as $y = a\left[f(x) + \frac{d}{a}\right]$. In this case, vertical translation by $\frac{d}{a}$ units is done before vertical stretch by factor of a.

A student may be tempted to argue that since addition is commutative, the order of translation and stretching could be reversed, i.e., vertical translation by d units occurring before vertical stretch by factor of a should give the same effect. This argument is incorrect. A counter-example is shown in **Table 1**, using the function shown in **Figure 2**.

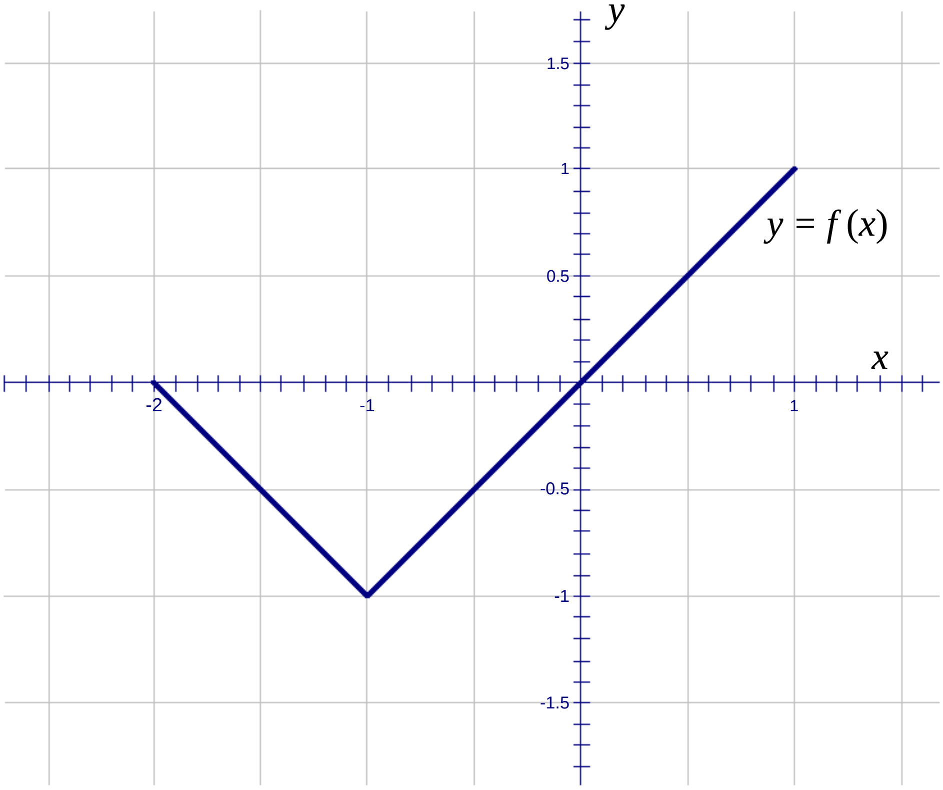

Figure 2

Function	Four specific pairs of coordinates of $f(x)$ in **Figure 2**			
$f(x)$	(–2, 0)	(–1, –1)	(0, 0)	(1, 1)
$af(x)$	(–2, 0)	(–1, $-a$)	(0, 0)	(1, a)
$af(x)+d$	(–2, d)	(–1, $d-a$)	(0, d)	(1, $d+a$)
$f(x)+d$	(–2, d)	(–1, $d-1$)	(0, d)	(1, $d+1$)
$a[f(x)+d]$	(–2, ad)	(–1, $ad-a$)	(0, ad)	(1, $ad+a$)

Table 1 An example illustrating that vertical translation and vertical stretch cannot interchange

Four pairs of coordinates from the function in **Figure 2** are shown in the 1st row of **Table 1**. The 2nd row is obtained by applying a vertical stretch of *a* units on $f(x)$ that affected the *y*-coordinates only. The 3rd row is the final result of $af(x)+d$.

In the 4th row, vertical translation by *d* units is applied to coordinates of $f(x)$. The result after applying vertical stretch by factor of *a* is shown in the 5th row.

Coordinates in the 5th row are certainly different from those in the 3rd row.

Thus, reversing order of vertical translation by *d* units followed by vertical stretch by factor of *a* would produce different result. Furthermore, it may be observed that

$$\begin{aligned} a[f(x)+d] &= af(x)+ad \\ &\neq af(x)+d \quad \text{if } a\neq 1 \end{aligned}$$

Note 2: On Transformations that only act on the Domain

In this section, we consider transformations such as $y=f(bx+c)$ where both horizontal stretch by factor of $\frac{1}{b}$ and horizontal translation by *c* units act on the domain of the original function $f(x)$. The image of $f(x)$ is not being acted on. Without loss of generality, we assume *b* and *c* to be positive.

Many students mistakenly think that $y = f(bx + c)$ involves first a horizontal stretch of *b* and then a horizontal translation of *c* units to the right for $c > 0$. On the contrary, many students are able to identify correctly that $y = af(x) + d$ involves a vertical stretch by factor of *a* and vertical translation up by *d* units if $d > 0$.

This is not surprising because the nature of the stretch matches the magnitude of *a*, where $a > 1$ and $a < 1$ represent dilation and compression respectively. Similarly, the direction of the vertical translation matches the sign of *d*, where $d > 0$ and $d < 0$ represent translation up and down respectively.

Alternatively, it may be interpreted that the direction of the vertical translation matches the addition/subtraction operation in $y = af(x) + d$ that, when $d > 0$, plus and minus signs represent translating up and down respectively.

Unfortunately, this logic does not apply to transformations that act on the domain of the original function $f(x)$. Most students resort to rote learning when dealing with $y = f(bx + c)$. Most of these students memorize the correct transformations without understanding. Moreover, most textbooks present transformations from graphical perspectives and offer very few explanations that facilitate deeper understanding. An explanation is attempted here.

Suppose $f(x)$ is transformed into $f(bx + c)$.Since both transformations in $y = f(bx + c)$ act only on the domain of the original function $f(x)$ but do not act on the image, there exists x' such that $y = f(bx' + c) = f(x)$.

Since $f(bx' + c) = f(x)$, then

$$bx' + c = x,$$

$$x' = \frac{x - c}{b}. \qquad \text{.................................(1)}$$

Equation (1) tells us how x' is obtained as horizontal translation from an arbitrary value *x*. In particular, (1) says that the arbitrary value *x* has to be translated horizontally *c* units to the left (assuming that $c > 0$) and then stretched horizontally by factor of $\frac{1}{b}$. The plus sign and minus sign in (1) represent horizontal translation to the right and left respectively. The horizontal stretch in (1) is precisely the multiple of $(x - c)$.

Thus, similar logic that applies to vertical transformation can be applied to horizontal transformations albeit via (1). The order of transformations is according to the conventional order of operations.

The equation in (1) can also be expressed as

$$x' = \frac{x}{b} - \frac{c}{b}. \qquad \text{......................(2)}$$

Viewed as (2) and applying the conventional order of operations, the arbitrary value *x* has to be stretched horizontally by factor of $\frac{1}{b}$ first before translated horizontally $\frac{c}{b}$ units to the left.

To summarize, the correct order of transformations for $y = f(bx + c)$ is either

(i) a horizontal translation *c* units to the left (assuming that $c > 0$) and then a horizontal stretch by factor of $\frac{1}{b}$; or

(ii) a horizontal stretch by factor of $\frac{1}{b}$ and then a horizontal translation $\frac{c}{b}$ units to the left.

Note 3: On Reflections

Consider $y = -af(x)$ where $a > 0$. We wish to establish whether or not reflection about *x*-axis come before vertical stretch by factor *a*.

Applying the conventional order of operation, we can treat $-a$ as $(-1)(a)$. Note that multiplication is commutative. Thus, we can conclude that reflection about x-axis and vertical stretch by factor a can interchange order. In other words, either reflection about x-axis or vertical stretch by factor a can occur first.

Here is another example. The function in **Figure 2** is transformed via stretching then reflection (rows 2 and 3) and reflection then stretching (rows 4 and 5).

Function	Four specific pairs of coordinates of $f(x)$ in **Figure 2**			
$f(x)$	(–2, 0)	(–1, –1)	(0, 0)	(1, 1)
$af(x)$	(–2, 0)	(–1, –a)	(0, 0)	(1, a)
$-af(x)$	(–2, 0)	(–1, a)	(0, 0)	(1, –a)
$-f(x)$	(–2, 0)	(–1, 1)	(0, 0)	(1, –1)
$a[-f(x)]$	(–2, 0)	(–1, a)	(0, 0)	(1, –a)

Table 2 An example illustrating interchange of reflection about x-axis and vertical stretch

The results in the 3rd row and the 5th row in **Table 2** are identical. It is observed that reflection about x-axis and vertical stretch can interchange order and arrive at the same result for our function in **Figure 2**.

Similarly, consider $-b$ in $y = f(-bx)$, where $b > 0$, as $(-1)(b)$. Note that multiplication is commutative. Reflection about y-axis and horizontal stretch by factor of $\frac{1}{b}$ may interchange order.

Note 4: More on Transformations involving $f(-bx+c)$

Consider the order of transformations in $y = f(-bx+c)$ where $b, c > 0$. Apply the method of $y = f(-bx'+c) = f(x)$ to obtain

$$-bx'+c = x$$

$$x' = -\frac{x-c}{b} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

In Equation (3), the transformation occurs in the following order: horizontal translation c units to the left, followed by horizontal stretch by factor of $\frac{1}{b}$, and lastly reflection about y-axis. We can treat this as the conventional interpretation.

In fact, from previous section we know that reflection about y-axis and horizontal stretch can interchange order. There are actually 3! permutations of interpretations of the three transformations in $y = f(-bx + c)$. The other five interpretations are displayed in **Table 3**.

Expression	Interpretations of transformations
$x' = \frac{-(x-c)}{b}$	Horizontal translation c units to the left, reflection about y-axis, and horizontal stretch by factor of $\frac{1}{b}$.
$x' = \frac{-x+c}{b}$	Reflection about y-axis, horizontal translation c units to the right, and horizontal stretch by factor of $\frac{1}{b}$.
$x' = \frac{-x}{b} + \frac{c}{b}$	Reflection about y-axis, horizontal stretch by factor of $\frac{1}{b}$, and horizontal translation $\frac{c}{b}$ units to the right.
$x' = -\left(\frac{x}{b}\right) + \frac{c}{b}$	Horizontal stretch by factor of $\frac{1}{b}$, reflection about y-axis, and horizontal translation $\frac{c}{b}$ units to the right.
$x' = -\left(\frac{x}{b} - \frac{c}{b}\right)$	Horizontal stretch by factor of $\frac{1}{b}$, horizontal translation $\frac{c}{b}$ units to the left, and reflection about y-axis.

Table 3 Five other permutations of the three transformations in $y = f(-bx + c)$ where $b, c > 0$

Most students will probably find the results in **Table 3** above an over-kill. My recommendation is to keep things simple and stick only to the interpretation suggested by (3).

Conclusion

Now we are ready to tackle the order of transformations in $y = af(bx + c) + d$. We follow the conventional order of operations.

We begin with the horizontal transformations. According to results in previous sections there are six permutations of order of transformations. For simplicity, we use the convention that $x' = \frac{x - c}{b}$ consists of horizontal translation to the left by *c* units (for $c > 0$) and a horizontal stretch by factor of $\frac{1}{b}$.

The direction of the horizontal translation matches the sign of the operation between *x* and *c* in (1) where plus and minus means right and left respectively. The x' method allows students to employ the same logic they used in understanding transformations that only act on the image of the original function.

After the horizontal transformations above, we have vertical stretch by factor of *a* and lastly vertical translation by *d* units. Note also that the order of reflection about *x*-axis and vertical stretch can interchange, and the order of reflection about *y*-axis and horizontal stretch can also interchange.

Returning to our motivation questions, the answer to Q.1 is (b). We can establish in this case that the order of vertical and horizontal translations can interchange by simply using the order of operations.

We can use Q.1 as a hint to solve Q.2. Consider reflecting *g* about *x*-axis and then horizontally stretch it by factor of 2. From Note 3, we can also stretch *g* by factor of 2 before reflection about *x*-axis.

I set a modified Q.2 in a class test and many students thought incorrectly that it involved translating 5 units horizontally to the right. We can verify that the formulation for h is given by $h = -f\left(\frac{x}{2} - 4\right) - 2$. Applying the x' method, we have $x' = 2(x+4)$ which can be treated as translating 4 units horizontally to the right and a horizontal stretch by factor of 2 or a horizontal stretch by factor of 2 followed by horizontal translation of 8 units to the right. Either way, there is no horizontal translation of 5 units to the right.

Q.3 is not related to the Q.1 or Q.2 and I invite readers to work out the exact order of transformations.

Reference

Curriculum Development Council & Hong Kong Examinations and Assessment Authority (2007). *Mathematics Curriculum and Assessment Guide (Secondary 4 - 6)*. http://www.edb.gov.hk/FileManager/EN/Content_5999/math_final_e.pdf

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