Bargain Hunting in Grovetown

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Jack and Jill went up the hill to Grovetown. Business had not been robust since the introduction of the so-called *New Math*, and many shops offered substantial discounts. The first stop they made was at Panorama 360. This shop was located at the town centre, with straight roads going off in many directions. There was a big sign at the front of the shop. It said, "Big Deal, 50% off!"

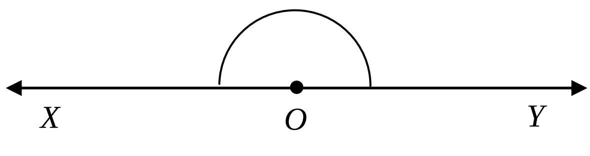
"What is the meaning of the name of your shop?" Jack asked the owner, who met them at the door.

"If you measure the angle between every pair of adjacent roads meeting at my shop and add up all these angles, the total will be 360°."

"Let's go in," said Jill.

The owner barred the door and handed them a piece of paper. He said, "Before you are allowed to enter any shop in Grovetown, you must first answer a skill-testing question."

The piece of paper has a diagram and the instruction: " OX and OY are two rays from a point O. What is the measure of $\angle XOY$ if these two rays form a straight line?"



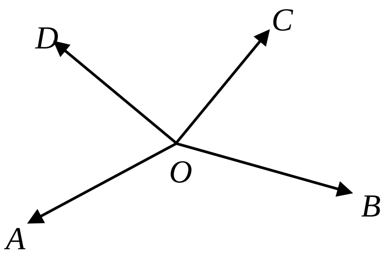
"That is easy," said Jill. "The angle on the other side of the line is exactly the same as $\angle XOY$. Since the two add up to 360°, each must be 50% of 360°, that is, 180°."

"Correct!" beamed the owner. "Moreover, if $\angle XOY = 180°$, then OX and OY form a straight line. Anyway, you may now enter my shop."

At the centre of the shop, they saw a billboard with a diagram and the instruction: " OA, OB, OC and OD are four rays from a point O. Buy two of the following and you will get the others for free.

(1) The rays OA and OC form a straight line.
(2) The rays OB and OD form a straight line.
(3) $\angle AOB = \angle COD$.
(4) $\angle BOC = \angle DOA$.

Combo I: (1) and (2).
Combo II: (3) and (4).
Combo III: (1) and (3), or (1) and (4), or (2) and (3), or (2) and (4)."



"Let me see," muttered Jack. "Suppose I buy Combo I. Then $\angle AOB + \angle BOC = 180°$ and $\angle BOC + \angle COD = 180°$. Ah ha, I now have (3), namely, $\angle AOB = \angle COD$."

"You can get (4) too, in more or less the same way. What if we buy Combo II instead?"

"Well, we have $\angle AOB + \angle BOC + \angle COD + \angle DOA = 360°$. So we will have $\angle AOB + \angle BOC + \angle AOB + \angle BOC = 360°$. In other words, $\angle AOB + \angle BOC = 180°$, and this means OA and OC form a straight line. So we do have (1)."

"We would also have (2). The only cases we have not yet considered are if we buy Combo III.," said Jill. "Which of the four options should we go for?"

"I think they are more or less the same. Let's buy (1) and (3), and go to another shop. We can work out later how we can get (2) and (4)."

"That will cost you two shillings," said the owner.

Jill forgot to bring any money and Jack had only a crown. However, the till was empty as they were the first customers in the shop that day, so the owner could not give change. Jack went down the hill and broke his crown at a money-changer. Then he came back and paid the exact amount.

The next shop was the Bermuda Triangle. This was the skill-testing question. "The vital statistics of triangle ABC are the length a of BC, the length b of CA, the length c of AB, the measure α of $\angle A$, the measure β of $\angle B$ and the measure γ of $\angle C$, where $0° < \alpha$, β, $\gamma < 180°$.

(a) If arbitrary values α, β and γ are given, does that always exist a triangle *ABC* such that $\angle A = \alpha$, $\angle B = \beta$ and $\angle C = \gamma$?

(b) What is the answer if arbitrary values are given for a different set of three vital statistics?

"Part (a) is easy," said Jack. "The triangle may be of any size, but there is always one such triangle."

Jill agreed, but the owner did not.

"How can that be?" Jack was puzzled.

"Oh!" said Jill. "In any triangle *ABC,* we must have $\alpha + \beta + \gamma = 180°$. If the given values are $\alpha = \beta = \gamma = 5°$, there is no such triangle."

"I see. So the answer is also negative if we are given the measures of two of the angles plus the length of one of the sides. This is because each of the two angles may be given the measure 120°."

"Surely," said Jill, "if arbitrary values a, b and c are given, there is exactly one possible triangle ABC."

"I disagree," said Jack. "You should have said at most one possible triangle ABC. If $a = b = 1$ and $c = 24$, we have no triangle."

"The only case left," said Jill, "is if we are given the measures of one of the angles plus the lengths of two of the sides. I am pretty sure the answer must be positive."

The owner said, "It is true that if the angle whose measure is given is between the two sides whose lengths are given, then there exists such a triangle. Moreover, it is unique, meaning that all such triangles have the same shape and size. We say that they are congruent to one another. However, for the other combinations of an angle and two sides, the answer is still negative. So your answer is not correct. Go home and figure that out for yourselves first and then come back."

"We are almost correct," pleaded Jack, "and we will figure it out later. Can you give us a different skill-testing question?"

"All right. In triangle ABC, if $AB = AC$, does it follow that $\angle C = \angle B$?"

"Yes," said Jill right away.

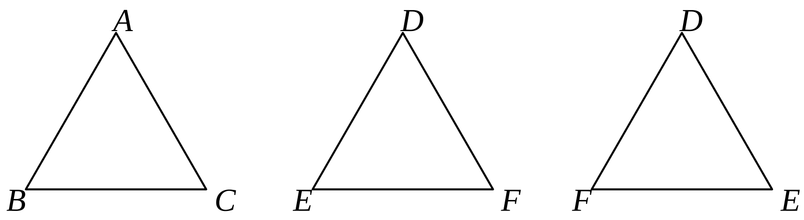
"You are right, but can you prove it?"

"Not yet," said Jack, " but I think I can now answer the last case left in first skill-testing question. There is no triangle with $a = 1$, $b = 2$ and $\alpha = 90°$."

"Good," said the owner, "but you should finish off the second skill-testing question, now that you have asked for it."

Jack and Jill huddled for a little while. Finally they found a neat proof.

"Make a copy of triangle ABC call it triangle DEF. Then $AB = DE$, $AC = DF$, $\angle A = \angle D$ and $\angle C = \angle F$. Now flip triangle DEF over into triangle DFE. Since $AB = AC$, we have $DF = AC = AB$, $DE = AB = AC$ and we still have $\angle A = \angle D$. From your remark on the last skill-testing question, triangles ABC and DFE are congruent to each other. It follows that $\angle B = \angle F$, and since $\angle F = \angle C$, we have $\angle B = \angle C$."



"Very nice," said a pleased owner. "We call triangle like ABC isosceles triangles, meaning that they have two equal sides. Moreover, if $\angle C = \angle B$, then $AB = AC$. You are now more than welcome to my shop."

At the centre of the shop, they saw a billboard with the instruction: "Let ABC and DEF be two triangles congruent to each other. Then the following six items are true. However, if you only buy three of them, you may get the others for free.

(1) $BC = EF$.
(2) $CA = FD$.
(3) $AB = DE$.
(4) $\angle A = \angle D$.
(5) $\angle B = \angle E$.
(6) $\angle C = \angle F$.

Combo I: (1), (2) and (3).
Combo II: (4), (5) and (6).
Combo III: (1), (2) and (6), or (2), (3) and (4), or (3), (1) and (5).
Combo IV: (1), (2) and (4), or (2), (3) and (5), or (3), (1) and (6), or (1), (2) and (5), or (2), (3) and (6), or (3), (1) and (4).
Combo V: (1), (5) and (6), or (2), (6) and (4), or (3), (4) and (5).

Combo VI: (1), (6) and (4), or (1), (4) and (5), or (2), (4) and (5), or (2), (5) and (6), or (3), (5) and (6), or (3), (6) and (4).

Buyers beware!"

"What do you think they mean by buyers beware?"

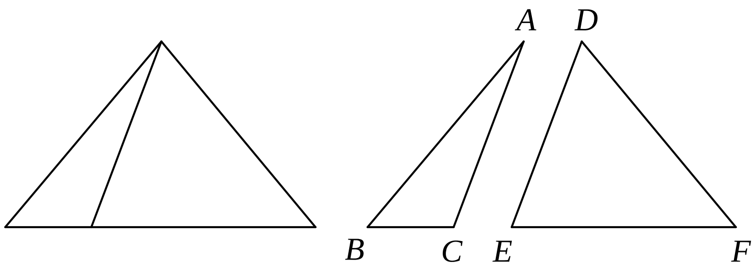
"In the last shop, they say you will get the others for free," Jill replied. "Here, they say you may get the others for free. So if we buy Combo II, we can only guarantee that triangles *ABC* and *DEF* have the same shape. They do not necessarily have the same size, so that we may not have any of (1), (2) and (3)."

"From the first skill-testing question," Jack said, "we can safely buy Combo III."

"I suspect from the second skill-testing question that we should not buy Combo IV," said Jill. "Aha! Here is why."

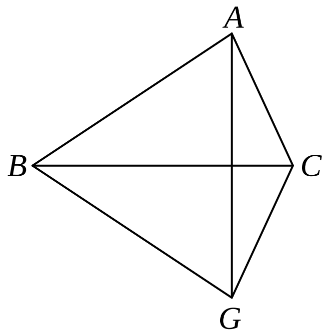
"Why?"

Jill drew a diagram on a piece of paper and showed it to Jack.



"Take any isosceles triangle and any point on its third side but not its midpoint," she explained. "Join this point to the opposite vertex and cut along this line segment to make two triangles. If we label the triangles as shown in my diagram, then we have (2), (3) and (5) but not (1), (4) or (6)."

"On the other hand, I think we can buy Combo I", Jack said, and showed his diagram to Jill. "Copy triangle *DEF* as triangle *GBC*, with the longest side *EF* coinciding with *BC*, and *G* and *A* being on opposite sides of BC. Now $GB = DE = AB$. By the second skill-testing question, we have $\angle BAG = \angle BGA$. Similarly, $\angle CAG = \angle CGA$, so that $\angle CAB = \angle CGB$. By the first skill-testing question, triangles ABC and GBC are congruent to each other, and hence to triangle DEF. It follows that we have (4), (5) and (6).



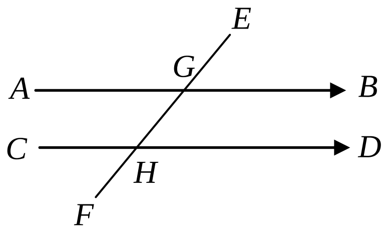
"This leaves Combos V and VI, which look alike," said Jill. "I think we can buy either of them, but we don't have much time. Let's buy (2), (5) and (6), and find out later if we have made a bad choice."

"You haven't," said the owner. "That will be three shillings."

"Now all my money is gone," lamented Jack. "However, let us visit one more shop, even if all we can do is window shopping."

The third shop they visited is the Parallel Universe. There was a big sign at the front. It said,"Almost Free, 75% off!"

The skill-testing question is on a piece of paper, with a diagram and the instruction: "A line EF cut two parallel lines AB and CD at G and H respectively. Of the eight angles formed at G and H, which are equal?"



Jill said, "That is easy. We have $\angle EGB = \angle HGA = \angle GHD = \angle FHC$ and $\angle EGA = \angle HGB = \angle GHC = \angle FHD$."

"Correct," said the owner. "We have $\angle EGB = \angle HGA$ and $\angle GHD = \angle FHC$ in any case. If these two sets of angles are equal, then AB and CD are parallel lines. The same can be said about the other four angles."

At the centre of the shop, they saw a billboard with the instruction: "Let $ABCD$ be a parallelogram whose diagonals intersect at E. Then the following eight items are true. However, if you only buy two of them, you may get the others for free.

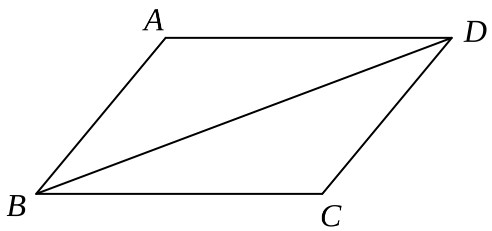
(1) AB and DC are parallel.
(2) AD and BC are parallel.
(3) $AB = DC$.
(4) $AD = BC$.
(5) $\angle A = \angle C$.
(6) $\angle B = \angle D$.
(7) $AE = CE$.
(8) $BE = DE$.

Combo I: (1) and (2).
Combo II: (3) and (4).
Combo III: (5) and (6).
Combo IV: (7) and (8).
Combo V: (1) and (3), or (2) and (4).
Combo VI: (1) and (4), or (2) and (3).
Combo VII: (5) and (7), or (6) and (8).
Combo VIII: (5) and (8), or (6) and (7).
Combo IX: (1) and (5), or (1) and (6), or (2) and (5), or (2) and (6).
Combo X: (1) and (7), or (1) and (8), or (2) and (7), or (2) and (8).
Combo XI: (3) and (5), or (3) and (6), or (4) and (5), or (4) and (6).
Combo XII: (3) and (7), or (3) and (8), or (4) and (7), or (4) and (8).

Buyers beware!"

Jack said, “Combo I is easy, because (1) and (2) are precisely the definition of a parallelogram.”

Jill drew a diagram and said, “I think we can buy Combo II. Just draw the diagonal BD, dividing the quadrilateral into triangles BAD and BCD. By (3) and (4), we have $AB = DC$ and $AD = BC$. Since $BD = BD$, we can go back to Bermuda Triangle and cash in their Combo I, so that triangles BAD and BCD are congruent.”

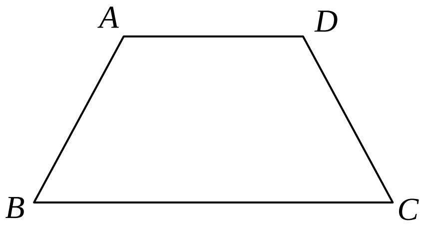


“Yes,” said Jack, “this means that $\angle ADB = \angle CBD$ and $\angle ABD = \angle CDB$. By the skill testing question here, we have (1) and (2), so that Combo II can be obtained by cashing in Combo I.”

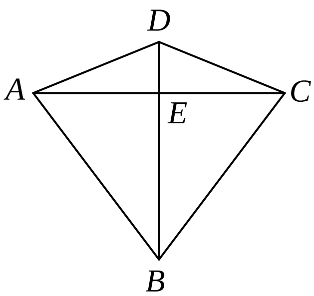
“Combo III and Combo IV look like good ones,” said Jill.

“Let us consider Combo V, the first mix-and-match case,” said Jack. “Suppose we buy (1) and (3). Using your diagram, we have $AB = DC$ and $BD = BD$. Since AB is parallel to DC, the skill testing question tells us that $\angle ABD = \angle CDB$. We can now go back to Bermuda Triangle and cash in their Combo III. The rest is easy.”

“On the other hand,” Jill remarked, “if we buy (2) and (3) instead and use the same reasoning, we will end up with Combo IV of Bermuda Triangle, which does not always work. Aha, here is a counter-example which shows that Combo VI here is no good. We have AD parallel to BC and $AB = DC$, and yet $ABCD$ is not a parallelogram.”



"Combo VII is no good either," Jack exclaimed. "Your counter-example has an axis of symmetry passing through the midpoints of opposite sides. Mine has an axis of symmetry passing through opposite vertices. Here we have $\angle A = \angle C$ and $AE = CE$, and yet $ABCD$ is not a parallelogram."

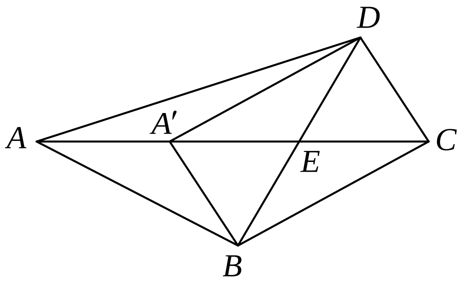


"This shop is a dangerous place," Jill shivered. "Makes you wonder if there are any more good combos here. The next one looks bad."

"In Combo VIII, $\angle A = \angle C$ and $BE = DE$," observed Jack. "If we also have $AE = CE$, then $ABCD$ is a parallelogram by Combo IV. So to create a counter-example, we may as well take $AE > BE$."

"Then $\angle A$ must be smaller than $\angle C$. So Combo VIII is good after all."

"Not so fast," cautioned Jack. "We must prove that we indeed have $\angle A < \angle C$. Aha! Let us take A' to be the point on AE such that $A'E = CE$. Then $A'BCD$ is a parallelogram."



"This means that $\angle BCD = \angle BA'D$," said Jill. "Surely, $\angle BA'D > \angle BAD$."

"It surely looks right, but why?" asked Jack.

"I see," said Jill excitedly. "We have $\angle BA'C = 180° - \angle BA'A = \angle ABA' + \angle BAC > \angle BAC$. Similarly, $\angle DA'C > \angle DAC$, so that indeed $\angle BA'D > \angle BAD$."

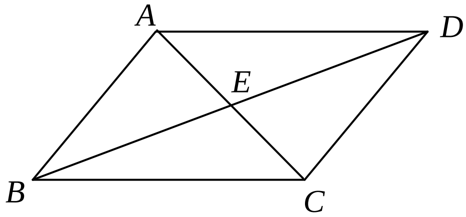
"So we will have a contradiction if $ABCD$ is not a parallelogram," said Jack.

"So Combo VIII is indeed a good one."

"A very nice of collaboration," beamed the owner.

"This restores my confidence in this store somewhat," Jill conceded. "I am sure Combo IX is also good, but what about Combo X?"

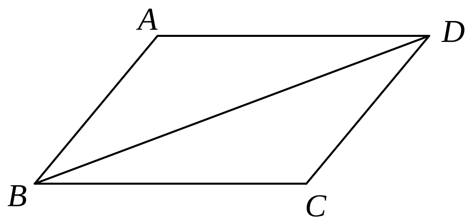
Jack drew a diagram and said, "Let us say we buy (1) and (7). Sine AB is parallel to DC, we know from the skill testing question here that $\angle BAE = \angle DCE$. From Panorama 360, we know that $\angle BEA = \angle DEC$. Along with $AE = CE$, triangles BAE and DCE are congruent by Combo V in Bermuda Triangle."



"Yes," said Jill. "This means we have $AB = DC$, so that Combo X follows from Combo V here. We only have two more to go."

"Take your time," said the owner. "These two are not so easy."

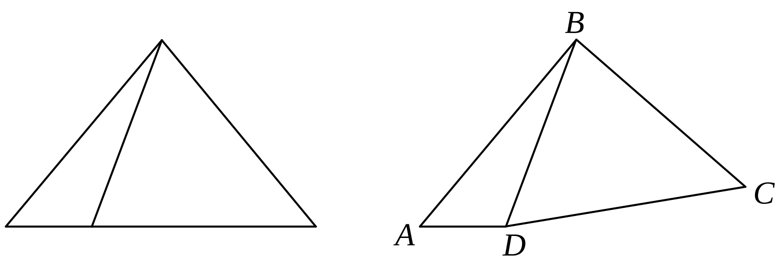
"Combo XI is easy," insisted Jack. "Suppose we buy (3) and (5). Then $AB = DC$, $\angle A = \angle C$ and $BD = BD$. So triangles BAD and DCB are congruent, and everything else follows."



"No," exclaimed Jill. "You are using Combo IV in Bermuda Triangle, but that one is no good!"

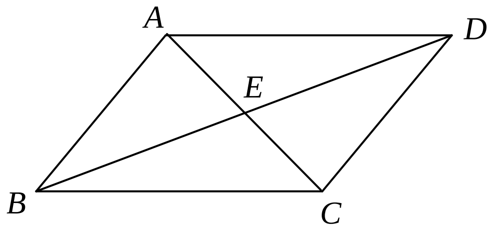
"You are absolutely right," admitted Jack. "I still have your counter-example for Combo IV in Bermuda Triangle. Perhaps we can get from this a counter-example to Combo XI here."

"That is a brilliant suggestion," said Jill, and quickly drew a diagram. "Just flip the bigger piece over and call it triangle BCD. Then glue it back to the smaller piece as shown here. We have $AB = DC$ and $\angle A = \angle C$, but $ABCD$ is not a parallelogram."

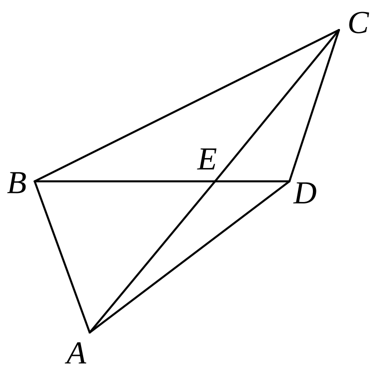


"Is Combo XII good?" Jack asked.

"I think so," replied Jill. "Suppose we buy (3) and (7). Then $AB = DC$, $AE = CE$ and $\angle AEB = \angle CED$. So triangles AEB and CED are congruent, and the rest is easy."



"Not so fast," said Jack, "because now you are using Combo IV in Bermuda Triangle. If we make the smaller piece of your counter-example CED and turn the bigger piece around and make it AEB, we have (3) and (7), but $ABCD$ is not a parallelogram. Combo XII is no good either!"



"I think I have had enough bargain hunting for a day," said Jill. "I need to relax."

"Let's go to a sauna and let off some steam," suggested Jack.

"Good idea," said Jill. "Then we will get on the trampoline and do some tumbling after."

Epilogue

Later at home, Jack took care of unfinished business.

Combo III of Panorama 360.

Given: (1) The rays OA and OC form a straight line. (3) $\angle AOB = \angle COD$.

To prove: (2) The rays OB and OD form a straight line. (4) $\angle BOC = \angle DOA$.

Proof:

By (1), $\angle AOB + \angle BOC = 180°$. Combined with (3), $\angle COD + \angle BOC = 180°$. This yields (2), which in turn yields (4).

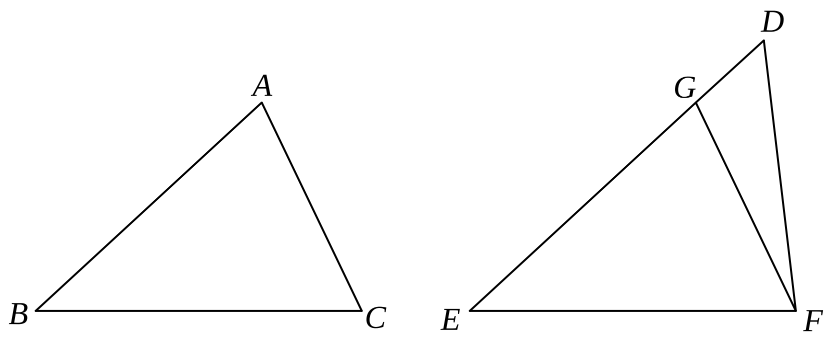
Combo V of Bermuda Triangle.

Given: (1) $BC = EF$, $\angle B = \angle E$ and (6) $\angle C = \angle F$.

To prove: ABC and DEF are congruent triangles.

Proof:

There is nothing to prove if we also have $AB = DE$. Suppose this is not the case. By symmetry, we may assume that $AB < DE$. Take G to be the point on DE such that $GE = AB$. Then triangles ABC and GEF are congruent, so that $\angle ACB = \angle DFE$. By (6), $\angle ACB = \angle DFE$. Hence $\angle GFE = \angle DFE$, which is clearly impossible.



Combo VI of Bermuda Triangle.

Given: (1) $BC = EF$, (6) $\angle C = \angle F$ and (4) $\angle A = \angle D$.

To prove: ABC and DEF are congruent triangles.

Proof:

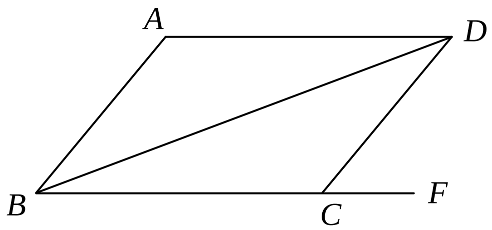
We have $\angle A + \angle B + \angle C = 180° = \angle D + \angle E + \angle F$. It follows from (6) and (4) that $\angle B = \angle E$. The desired result now follows from Combo V above.

<u>Combo III of Parallel Universe.</u>

Given: (5) $\angle A = \angle C$. (6) $\angle B = \angle D$.

To prove: $ABCD$ is a parallelogram.

Proof:



We have $\angle A + \angle B + \angle C + \angle D = \angle A + \angle ABD + \angle ADB + \angle C + \angle CBD + \angle CDB = 360°$. By (5) and (6), $\angle B + \angle C = 180° = \angle C + \angle DCF$. Hence $\angle B = \angle DCF$, so that AB is parallel to DC. Similarly, AD is parallel to BC.

<u>Combo IV of Parallel Universe.</u>

Given: (7) $AE = CE$. (8) $BE = DE$.

To prove: $ABCD$ is a parallelogram.

Proof:

We have $\angle AEB = \angle CED$. Hence triangles AEB and CED are congruent.

<u>Combo XI of Parallel Universe.</u>

Given: (1) AB parallel to DC. (5) $\angle A = \angle C$.

To prove: $ABCD$ is a parallelogram.

Proof:

By (1), $\angle ABD = \angle CDB$. Hence triangles BAD and DCB are congruent.