A Discussion of Q19(b)(iii) in the HKDSE Practice Paper Mathematics Compulsory Part Paper 1

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Background

The first Hong Kong Diploma of Secondary Education Examination (HKDSE) will be held this year. For candidates to get familiarized with the requirements of the new public examination, a series of practice papers were produced and disseminated by HKEAA. All candidates from day schools would be arranged to attempt the papers. Looking at the practice paper of Mathematics Compulsory Part Paper 1, most students got no marks from the last question, Q19(b)(iii), which is shown as follow:

(HKEAA, 2012a) Q19(b)(iii) *The manager of the firm claims that the total revenue made by the firm will exceed the total amount of investment. Do you agree? Explain your answer.*

In this paper, the answer and marking of the question will be discussed.

Students' Answer

The total amount of investment in the long run

$$=\frac{4000000}{1-0.64}\approx \$11111111.1$$

$>\$10000000=$ The total revenue in the long run (from (b)(ii))

$\therefore$ I do not agree with the claim of the manager.

Discussion

Similar answers were observed in many scripts and these students received no marks because they failed to compare the total amount of investment with the total revenue year by year as suggested in the provisional marking scheme

(HKEAA, 2012b). Full marks would only be given to those who correctly answered the question using the standard method suggested. In this paper, it is recommended that 1 mark should be given, instead of no mark.

Giving two sequences $\{T_1(n)\}$ and $\{T_2(n)\}$, it is naturally for us to compare the sequences term by term. Generally, there is no guarantee $T_1(n) \geq T_2(n)$ or $T_1(n) \leq T_2(n)$ for all positive integers n . However, if the general terms of the sequences are known, comparison would be possible. Looking at Q19(b)(iii), the amount of investment and the revenue each year form two geometric sequences. And the total amount of investment and the total revenue after n years form two geometric series. Thus, the properties of geometric sequences and series can be used in answering Q19(b)(iii), including the properties of infinite geometric series.

Markers would argue that there might be a particular value of n , such that the total revenue after n years might be greater than the total amount of investment after n years, even if the total amount of investment in the long run is greater than the total revenue in the long run. However, for the case in Q19(b)(iii), such worry is unnecessary because of the following arguments.

Let $\{T_1(n)\}$ and $\{T_2(n)\}$ be two positive geometric sequences such that $0 < T_1(1) < T_2(1)$, and with their common ratios $1 > r_1 > r_2 > 0$. Intuitively, we see that although $0 < T_1(1) < T_2(1)$, if $T_1(k) > T_2(k)$ for some positive integer k , then we have $T_1(n) > T_2(n)$ for all $n \geq k$. Students were able to see such phenomenon when investigating the properties during lessons, in particular, the number of point of intersection of the graphs of two strictly decreasing exponential functions in the same coordinate plane. Similarly, students might intuitively think that for any two corresponding geometric series $\{S_1(n)\}$ and $\{S_2(n)\}$, $0 < S_1(1) < S_2(1)$. If $S_1(k) > S_2(k)$ for some positive integer k , then we have $S_1(n) > S_2(n)$ for all $n \geq k$. Students were able to see such phenomenon when investigating the properties during lessons, in particular, the number of point of intersection of the graphical representations of two strictly increasing functions in the same coordinate plane. They would like to answer Q19(b)(iii) by just considering the sum to infinity according to their intuition, which is not explicitly stated in the syllabus but could be developed from studying the NSS curriculum. The point here is that if a student utilizes such intuition to solve a problem in an examination, should zero mark be given?

It is something that is not stated explicitly but we, as well as students, may think that it is true, at least intuitively. Now, I try to give a proof.

Let $S_1(n) = a_1 + a_1r_1^{\,1} + ... + a_1r_1^{\,n-1}$ and $S_2(n) = a_2 + a_2r_2^{\,1} + ... + a_2r_2^{\,n-1}$ be two geometric series with $1 > r_1 > r_2 > 0$ and $0 < S_1(1), S_2(1)$. Note that S_1 and S_2 are two convergent and strictly increasing sequences.

Theorem. Given $S_1(1) = a_1$, $S_2(1) = a_2$, $0 < a_1 < a_2$, and $1 > r_1 > r_2 > 0$. If $S_1(k) > S_2(k)$ for some positive integer k, then $S_1(n) > S_2(n)$ for all $n \ge k$.

Proof. Suppose there exists a positive integer k such that $S_1(k-1) < S_2(k-1)$ and $S_1(k) > S_2(k)$. By subtracting the two inequalities, we have $S_1(k) - S_1(k-1) > S_2(k) - S_2(k-1)$. Simplifying the expression, we have $a_1r_1^{\,k-1} > a_2r_2^{\,k-1}$, and so $\dfrac{r_1^{\,k-1}}{r_2^{\,k-1}} > \dfrac{a_2}{a_1}$.

Assume there exists another nonnegative integer r such that $S_1(k+r) > S_2(k+r)$ and $S_1(k+r+1) < S_2(k+r+1)$. By subtracting the two inequalities, we have $S_2(k+r+1) - S_2(k+r) > S_1(k+r+1) - S_1(k+r)$. Simplifying the expression, we have $a_2r_2^{\,k+r} > a_1r_1^{\,k+r}$, and so $\dfrac{a_2}{a_1} > \dfrac{r_1^{\,k+r}}{r_2^{\,k+r}}$.

Combining with the two inequalities, we have $\dfrac{r_1^{\,k-1}}{r_2^{\,k-1}} > \dfrac{a_2}{a_1} > \dfrac{r_1^{\,k+r}}{r_2^{\,k+r}}$, which is impossible as $\dfrac{r_1}{r_2} > 1$ and $k-1 < k+r$ for all nonnegative integer r. ❑

Corollary 1. Given $0 < a_1 < a_2$ and $0 < r_2 < r_1 < 1$. If $S_1(\infty) > S_2(\infty)$, then there exists a positive integer k such that $S_1(n) > S_2(n)$ for all $n \ge k$.

Proof. $0<a_1<a_2$ means that $S_1(1)<S_2(1)$. Suppose $S_1(\infty)>S_2(\infty)$, then, there must exist a positive integer k such that $S_1(k)<S_2(k)$ and $S_1(k+1)>S_2(k+1)$ because both are strictly increasing. The result follows by the theorem immediately. ❑

Corollary 2. Given $0<a_1<a_2$ and $0<r_2<r_1<1$. If $S_1(\infty)<S_2(\infty)$, then $S_1(n)<S_2(n)$ for all positive integers n.

Proof. Suppose the contrary is correct. If there exists a positive integer k such that $S_1(k)>S_2(k)$, by the corollary 1, we have $S_1(n)>S_2(n)$ for all $n\geq k$, so that $\lim_{n\to\infty} S_1(n)>\lim_{n\to\infty} S_2(n)$, which contradicts to the condition given. ❑

Corollary 3. If $S_1(\infty)>S_2(\infty)$ with $0<a_2<a_1$, then $S_1(n)>S_2(n)$ for all positive integers n.

Proof. It is trivial by directly comparing the two series term by term. ❑

The table below shows the specific details of the two geometric series in Q19(b)(iii) and the conclusion follows immediately by corollary 2[1].

Total Revenue $S_1(n)$	$a_1=2000000$	$r_1=0.8$
Total Amount of Investment $S_2(n)$	$a_2=4000000$	$r_2=0.64$

We see that the above mentioned candidates were mathematically correct but there was no support argument. Thus, I personally recommend that 1 mark can be given to these candidates giving the correct conclusion by considering the sum to infinity only.

Conclusion

According to the level descriptors in EDB (2009), candidates will not receive high marks in Q19(b)(iii), by just showing the computations of the sum to infinity without any explanations. The method purposed by HKEAA is

1 There is an alternative proof stated in the end note of this article.

suitable for a more general situation. But in public examination, candidates might use the properties of geometric sequences or series to answer the question intuitively. Therefore, it is recommended that in that part, 1 mark can be given because the argument is mathematically correct and the idea could be derived from the NSS curriculum. If similar cases arise in future, it is reasonable to give some marks. Finally, if candidates can come up with the conclusion by giving the proof of corollary 2, full marks should be awarded.

Note: Alternate proof for Corollary 2

As $S_1(\infty)=\frac{a_1}{1-r_1}<\frac{a_2}{1-r_2}=S_2(\infty)$,

we have $\frac{S_1(\infty)}{S_2(\infty)}=\frac{\frac{a_1}{1-r_1}}{\frac{a_2}{1-r_2}}<1$.

Hence, we have $\frac{S_1(n)}{S_2(n)}=\frac{\frac{a_1}{1-r_1}(1-r_1^n)}{\frac{a_2}{1-r_2}(1-r_2^n)}<\frac{1-r_1^n}{1-r_2^n}<1$

for all positive integers n , as $0<r_2<r_1<1$. ❑

References

EDB (2009). *Draft Level Descriptors (Compulsory Part)*, p.13.

HKEAA (2012a). *Practice Paper Mathematics Compulsory Part Paper 1 Question-Answer Book*, p.22.

HKEAA (2012b). *Provisional Marking Scheme of Practice Paper Mathematics Compulsory Part Paper 1*, p.16.

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