An important but overlooked theorem in 3-D Geometry

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Let's study the following problem:

Problem In Figure 1, it shows a cuboid $ABCDEFGH$ in which $FG = 12$, $GH = 9$ and $DH = 8$. Find the length of CE.

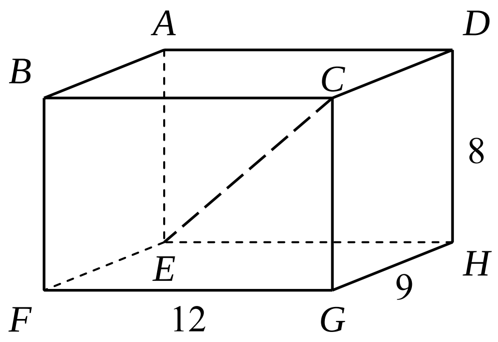

Figure 1

The solution can be given in two steps.

Step 1: Join EG. By Pythagoras' Theorem, $EG = \sqrt{12^2 + 9^2} = 15$.

Step 2: Then by Pythagoras' Theorem again, $EC = \sqrt{15^2 + 8^2} = 17$.

This question is an application of Pythagoras' Theorem to a three dimensional situation. In his famous book *How to Solve It*, Pólya also used a similar example to demonstrate how to apply his problem-solving strategy to find the diagonal of a rectangular parallelepiped. I expect this is not a difficult task for many teachers and students. However, when I finished explaining the method and solution to my students, sometimes some students would come out and ask me why we could apply Pythagoras' Theorem in Step 2.

Pythagoras' Theorem states, "In a right-angled triangle, the square of the side opposite to the right angle is equal to the sum of the squares of the sides adjacent to the right angle." Very often, students did not understand how Pythagoras' Theorem could be applied in Step 2 because they could not see why ΔCEG was a right-angled triangle or why $\angle CGE = 90°$.

In the first few years of my teaching, my answer to those students was, "It is *obvious*. As CG is perpendicular to the base $EFGH$ and EG is a line on $EFGH$, CG must be perpendicular to EG and $\angle CGE = 90°$."

However in one year, a student asked me a follow-up question which made me speechless. He said, "I know $ABCDEFGH$ is a cuboid. But which part of the question tells me that CG is perpendicular to $EFGH$?"

By definition, a cuboid is a solid figure bounded by six rectangles. This means $\angle CGF = \angle CGH = \angle FGH = 90°$ but *we have not been given that* $\angle CGE = 90°$ nor CG is perpendicular to the plane $EFGH$! As $\angle CGE$ is not given to be a right angle, it is quite possible that some students would be unable to observe this fact. The challenge is how can I help them seeing this?

It is not difficult to prove that two opposite faces of a cubold are rectangles congruent to each other. Therefore the given condition in this question is equivalent to say that there are 3 pairs of congruent rectangles. When I look back at my solution to the problem, I find that I have used the fact $EFGH$ being a rectangle in Step 1. That is to say, Pythagoras' Theorem can be applied because $\angle EFG$ is a right angle. How about Step 2? Does this mean that we have to make use of the facts that $BFGC$ and $CGHD$ are rectangles to prove $\angle CGE = 90°$?

The answer is yes. The key was stated in Proposition XI.4 of Euclid's *Elements*. Here I quote the proposition as follows:

THEOREM If a straight line is perpendicular to two straight lines which cut one another at their common point of intersection, it will also be perpendicular to any line lying on the same plane as the two straight lines and passing through that common intersection point.

According to Figure 1, $CG \perp FG$ and $CG \perp HG$ as $BFGC$ and $CGHD$ are rectangles. Then by the above theorem, we can conclude that $CG \perp EG$ as EG is a line lying on the same plane as FG and HG and passing through G. Therefore $\angle CGE = 90°$ and we can apply Pythagoras' Theorem to ΔCEG to find CE in Step 2! Thus, the above theorem is very important. I can use this result to answer the follow-up question raised by my student. Of course, we still need to prove it. In fact, Euclid had already given a proof when this theorem was written 2300 years ago.

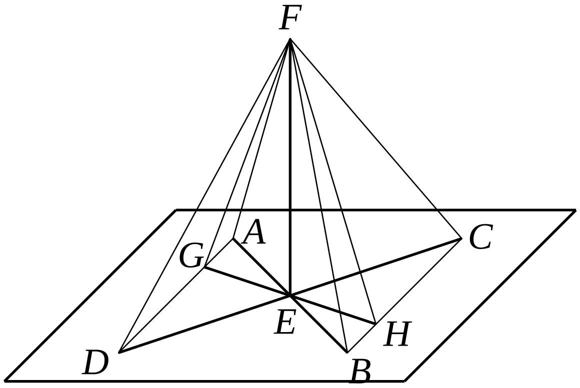

Figure 2

In Figure 2, suppose $FE \perp AB$ and $FE \perp CD$ at E. GH is any line lying on the same plane as $ABCD$ and passing through E. We are going to show $FE \perp GH$.

Without loss of generality, we may let $AE = BE$, $CE = DE$, G lies on AD and H lies on BC. Then the following 7 pairs of triangles are congruent to each other: $\Delta EAD \cong \Delta EBC$ (S.A.S.), $\Delta EAG \cong \Delta EBH$ (A.S.A.), $\Delta FEA \cong \Delta FEB$ (S.A.S.), $\Delta FED \cong \Delta FEC$ (S.A.S.), $\Delta FAD \cong \Delta FBC$ (S.S.S.), $\Delta FAG \cong \Delta FBH$ (S.A.S.) and $\Delta FEG \cong \Delta FEH$ (S.S.S.). From the last pair of congruent triangles, we can conclude that $\angle FEG = \angle FEH$. Hence, by definition, $FE \perp GH$. (Q.E.D.)

I must admit that the above theorem, despite its importance, is complicated. Not every student can understand. Maybe, this is the reason why Pólya had evaded this theorem (or maybe he had overlooked it) when he was presenting his problem-solving strategy in his famous book. However, we cannot deny that there are students who cannot see $\angle CGE = 90°$ in Figure 1. So, how can we help them?

In order to make it easier for my students to grasp the concept of how a straight line is perpendicular to a plane, I have given following demonstration in my lesson for a few years.

I prepared a foam board and a piece of bamboo stick beforehand. In class, I deliberately inclined the bamboo stick on the foam board at an angle other than 90° (see Photo 1). Students were asked if they thought the stick was perpendicular to the foam board. Students were usually confused by my act but, after some discussion, some would point out that it was impossible to describe an angle with just a line formed by the stick and without any lines given on the foam board (i.e. the plane). So, in order to "satisfy their needs", a line was then drawn on the foam board (as shown in Photo 2).

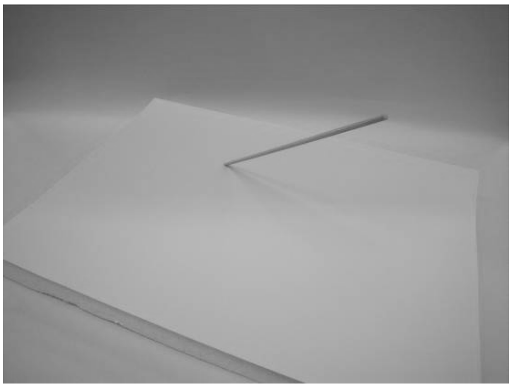

Photo 1: A bamboo stick inclining on a foam board

Photo2: A line drawn on the foam board forming an angle with the stick

The angle constructed was not a right angle, of course. Then I asked the students if we could claim that every line on the plane passing through the same intersection point would not be perpendicular to the stick.

Answer is negative. I could draw another line on the foam board so that it formed a right angle with the stick (as shown in Photo 3). At this moment, usually there were students pointing out that the rest of the lines on the plane were not perpendicular to the stick. However, if there was one more line on the foam board which was also perpendicular to the stick, then the stick would be perpendicular to the foam board (as shown in Photo 4).

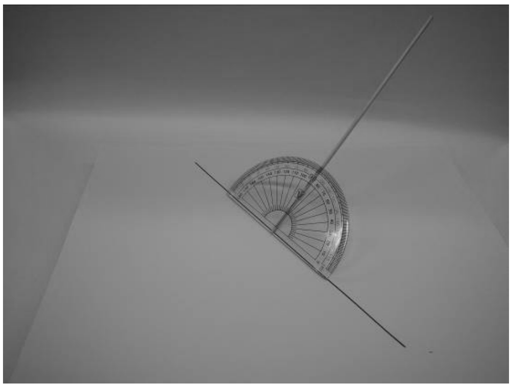

Photo 3: A line on the foam board is perpendicular to the stick.

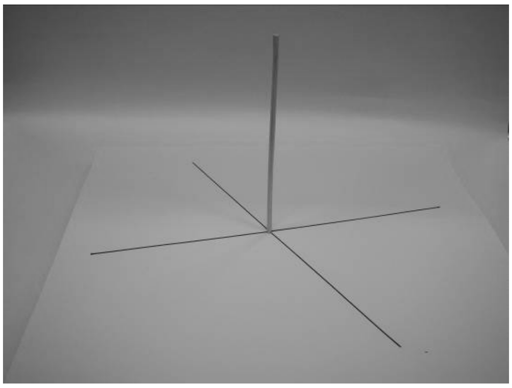

Photo 4: The stick is perpendicular to two lines on the board.

Finally, we could draw the following conclusion: If a bamboo stick was perpendicular to any two lines on a foam board, then it would be perpendicular to every line on the plane passing through the same intersection point. In fact, this is Preposition XI.4 of Euclid's *Elements*. The proof by Euclid would be given to those more able students for their reference. I think the above intuitive approach would be rigorous enough for general students.

To let students know how the above theorem was applied, one more line (as shown in bold letters) would be added to the solution of the above problem:

Since $\angle EFG = 90°$, $EG = \sqrt{12^2+9^2} = 15$ by Pythagoras' Theorem. Since $CG \perp FG$ and $CG \perp HG$, $CG \perp EG$, i.e. $\angle CGE = 90°$. Therefore, $EC = \sqrt{15^2+8^2} = 17$ by Pythagoras' Theorem.

Most students accept this presentation and it is believed that students would have a better concept about three-dimensional geometry.

References

Pólya, G. (1945). *How to Solve It*. New Jersey: Princeton University Press.

Heath, T.L. (1956). *The Thirteen Books of Euclid's Elements*. New York: Dover.

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