Inverse Relation Between Addition and Subtraction: How Well Can Primary Students Make Use of It?

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Background

The learning of arithmetic does not only mean knowing the computational skills, but also includes the understanding of the interrelation between the operations (Gilmore & Bryant, 2008). Improvement of conceptual understanding is believed to raise the performance of arithmetic calculation and students can be more flexible rather than mechanical in their calculation. For example, when calculating 234 + 398 – 395, a student may apply the addition-subtraction inverse principle to + 395 – 395 and see the original expression as simple as 234 + 3 (Baroody, Torbeyns, & Verschaffel, 2009). This leads to the difference between meaningful use of different strategies (adaptive expertise) and rote application of procedures (routine expertise) (Gilmore & Bryant, 2008). Working in the context of Hong Kong where students are usually found to perform well in arithmetic, the authors are concerned with their conceptual understanding of addition-subtraction inverse principle as well as the related strategies in handling computational problems. The addition-subtraction inverse principle means that adding a number b to another a can be undone by subtracting the same number b from the sum, and vice versa (i.e. $a + b - b = a$ and $a - b + b = a$) (Baroody et al., 2009).

Apart from the kind of computational strategies that would bring some convenience and advantages to mental calculation, there is greater significance

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* This paper is based on a piece of research conducted by a mathematics teacher and the subsequent report. With the consensus of the teacher, the identity of the teacher is not disclosed in the best interests of the school where the teacher is teaching. This note serves to recognize the major contribution of the anonymous teacher researcher.

of the inverse principle in relation to children's conceptual development. Piaget, cited in Nunes, Bryant, Hallett, Bell, and Evans (2009, p. 61), has linked the understanding of inverse principle "to [the more general] idea of reversibility, which is the ability to cancel the effect of a transformation to an object or a set of objects by imagining the opposite transformation to the material in hand." In view of the importance of the understanding of inverse relation to the learning addition and subtraction, there have been recent studies on the subject (Baroody et al., 2009).

According to the research done by Gilmore and Bryant (2008), in general, the children could understand the relationship between addition and subtraction, and handle quite well the *transparent inverse problems*, i.e. problems in which the inverse relation is obvious. They also concluded that individual difference was noticeable among children on their conceptual understanding and computational skill in arithmetic, which meant that some children could not connect the arithmetic principles with relevant calculation strategies to gain advantages. Their research leads to the consideration of how to link up the conceptual and procedural knowledge in children's development and raises the question about students' difficulties with the principle of inversion (Nunes et al., 2009).

Being concerned with both conceptual and procedural knowledge of primary students in arithmetic, we focus on how they use the addition-subtraction inverse principle in transparent and non-transparent inverse problems. In particular:

- How well do the students understand the principle of inversion?
- How well can they identify the related problems and apply the principle?
- What factors are affecting their identification of the inverse related problems?
- What are the differences, if any, among different Grades 2, 3 and 4?

Instruments and Sampling

Evidence for a response to the above questions is largely based on a written test which consists of a total of 14 questions in two sections. Section A (10 questions) involves calculations with actual numbers for which students can either perform step-by-step computation or use shortcut based on the inverse principle. In this written test, students were allowed to jot down on the question paper, if they wanted, their calculation work (e.g. the column arithmetic) which would provide evidence of their strategies. The test includes transparent inverse problems ($a + b - b = a$, $a - b + b = a$, or $a + b - a = b$) that are matched with a control problem in the form of $a + a - b$ to which students may misapply the inverse principle. These questions are providing good indicators of students' basic understanding of inversion (Gilmore & Bryant, 2008), Section A also includes four-term and five-term non-transparent inverse problems which call for more of students' comprehension of the situation and transformation of the numbers in order to take advantage of their knowledge of inversion and thus reveal how well, or flexibly, they apply their knowledge of inversion (Gilmore & Bryant, 2008). Section B of the written test involves calculation with only symbols (e.g. ○ – ◇ + ◇ = ?). Students were allowed to cross out a question if they did not know how to do it. This might prevent them from guessing the answers and raise the likelihood of tapping their true ability.

To suit the ability of the students at different grade levels, two versions of the written test were set. While the version for Grades 3 and 4 includes calculation with 3-digit numbers, the other for Grade 2 includes that with 2-digit numbers. To make performance results comparable, the question types in the two versions are parallel in that the questions for Grade 2 were simply modified by cutting the digits in the hundreds place of the numbers in the questions for Grades 3 and 4, e.g., 459 + 175 – 175 in Grades 3 and 4 was changed to 59 + 75 – 75 in Grade 2.

In this research, two classes from each of Grades 2, 3 and 4 in a local primary school of Hong Kong were chosen to take part in the written test. Altogether 152 students completed the written test. There were 49, 48, and 55 students in Grades 2, 3 and 4 respectively. All the classes were mixed ability classes and, according to the general understanding at the school, it can be assumed that approximately each of low-level, medium-level and high-level ability students took up one-third of the sample.

In order to understand better the reasons behind students' actions, especially for those who used shortcut (i.e. a total absence of calculation work) in all or some of the inverse problems, 18 students, i.e. six students from each grade level, were sampled after the written test for face-to-face interviews. The interviewees were asked to explain their work. Within limited scope, this paper reports less on the interview results than on the responses in the written test.

Analysis

The purpose of Section A is not only to show how many correct answers the students have got, but also what strategies they have used. So, each question in Section A carries two marks, one mark for accuracy and one mark for strategies. The mark for strategies was awarded if a student could identify the inverse related questions. For the questions in Section B that involved only symbols, students were not likely to perform routine computation. One mark is allocated for each of them. Scores on all the questions in the written test were then subject to analysis which was based on Item Response Theory (IRT) and the Rasch model (Bond & Fox, 2001). With the Rasch measurement which treated the scores as ordinal on a linear scale (Phillipson & Tse, 2007), both the performance of each student on different kinds of questions and the difficulty of different kinds of inverse related questions were analysed. The Rasch model, as a probabilistic model that produces the estimation of the students' ability levels and the items' difficulty on a linear scale (Phillipson & Tse, 2007), postulates that the difference between the ability of a person and the difficulty of an item defines the person's success in solving the item (Sideridis, 2007).

Results and Discussion

1. The students were more likely to have routine expertise

Among the students of all the three grade levels, the rough works, if any, jotted down in the written test are always column arithmetic and are symptomatic of mechanical calculation, i.e. direct input of numbers in a routine procedure without heeding the special relationship among the numbers. It is not suggestive of any shortcut strategy during their calculation. Students doing this have scored the lowest for the appropriate strategies.

2. The students did better in identifying transparent inverse problems

Although the recognition of inverse related problems was relatively more difficult for students than mechanical computation right from the start, students show a better performance in identifying the transparent inverse related situations than in the non-transparent inverse problems. According to the item measure of all students' calculating strategies (Table 1), Questions 1, 2 and 8 were the easiest inverse related problems for students to identify. In these three questions, the inverse principle is transparently applicable because the terms can be directly negated by each other. For Question 4, 5, 6 and 7, the second-term and the third-term cannot be negated directly because the numbers differ by 1 or 2. The students, who could associate the numerical expressions in question with the inverse principle, needed to reconstruct the numerical expression in each case by decomposing the relevant numbers so that two of the numbers could be negated by each other. Expectedly, the marks for strategies in these non-transparent inverse problems were not as good as those in the transparent ones. Furthermore, Questions 9 and 10 appear to have posed the most difficulty for the students in their recognition of the possible inverse relation involved. Like Questions 4 to 7, they are non-transparent; but they have five terms and the second term can only be negated by the total of the third and the fourth terms. In these two questions, there is the need for reconstruction of the expressions by combining the relevant numbers to set off a certain amount. For example, in Question 9, $653 + 279 - 170 - 109$ could be reconstructed as $653 + 279 - 279$ by combining "take away 170" and "take away 109" and inverse principle could be applied. Only a few students could recognize the possibility of using inverse principle in Questions 9 and 10, just as Rasch analysis results in their very high item measures.

The above findings show that students are less capable in recognizing the inverse problems in non-transparent situations where reconstruction of the numerical expressions is necessary if there would be any shortcut.

Item (Question)	Measure	Remarks
Q9: [P2] 53 + 79 – 70 – 9 [P3,4] 653 + 279 – 170 – 109	87.55	the most difficult question
Q10: [P2] 100 – 89 + 11 + 78 [P3,4] 321 – 189 + 111 + 78	86.72	
Q6: [P2] 104 – 78 + 80 [P3,4] 504 – 278 + 280	78.50	
Q7: [P2] 120 – 61 + 60 [P3,4] 820 – 461 + 460	75.15	
Q4: [P2] 54 + 59 – 60 [P3,4] 654 + 159 – 160	73.10	
Q5: [P2] 49 + 82 – 80 [P3,4] 349 + 582 – 580	69.79	
Q2: [P2] 78 – 39 + 39 [P3,4] 378 – 239 + 239	65.10	
Q8: [P2] 97 + 26 – 97 [P3,4] 297 + 326 – 297	65.10	
Q1: [P2] 59 + 75 – 75 [P3,4] 459 + 175 – 175	63.90	the least difficult question

Table 1: The item measures of strategies used in Questions 1 to 10 for all students (except the control question: Question 3)

3. The students were affected by the order of operations

The order of operations is shown to have an effect on the students' performance in identifying the inverse related situation. When two inverse related problems involved a negation between the second and the third terms, i.e. $a + b - b$ and $a - b + b$, students always identified better in the case where the addition came first. For example, the item measure of Question 2 (i.e. 378 – 239 + 239) was 1.2 higher than that of Question 1 (i.e. 459 + 175 – 175) (Table 1). A similar difference also occurred to the four non-transparent inverse problems (Questions 4 to 7), where the final term of each of the computational expressions differed from the second term by 1 or 2, e.g. 349 + 582 – 580 (Question 5) as contrasted with 504 – 278 + 280 (Question 6). Table 1 shows that the item measures of Questions 6 and 7 are both higher than those of Questions 4 and 5, which indicates an effect of the order of operations on students' recognition of inverse related element. Students can identify the inverse related element easier when the problem starts with a sum in hand (i.e. $a + b$) from which one of the addends is to be taken away ($-a$ or $-b$). On the other hand, difficulties arise when the problem starts by taking away a certain amount first (i.e. $a - b$, where $a > b$) and then putting it back ($+ b$). We interpret that students may not have a substantial idea about the difference $a - b$ which is not as concrete as a sum $a + b$ to start with. This contrast in item measures arises again, and is much stronger indeed, when the students are confronted with problems in Section B where they have to handle symbols but not numbers (see Table 2).

Question 11 is about "$a + b - b = a$". Students are expected to identify that "$+ b$" can be set off by "– b" and the original amount a resumes. Question 13 is similar but in a little different form "$a + b - a = b$" where the inverse elements, namely "a" and "– a" are not stuck together. Like that in the previous situations with particular numbers, this accounts for the difficulty of Question 13 as compared with Question 11. However, noteworthy is the observation that, notwithstanding the symbolic form which usually poses an obstacle to primary students, Questions 11 and 13 have turned out to be the easiest questions (item measures being 23.35 and 32.36 respectively) for the tested subjects at all the three grade levels. On the other hand, when the students are confronted with an inverse related problem in which the subtraction comes first and the resulting entity out of the first two terms thus becomes a difference instead of a sum, difficulties arise. Although Question 12 is the same as Question 11 in terms of the inverse relation between two adjacent terms, Table 2 shows the big difference in their difficulties to students (namely, the measure of 53.67 is more than doubled the measure 23.35). For Question 12 which now reveals an obvious difficulty, most of the students who got it wrong put a cross inside the brackets, claiming that it cannot be solved. We interpret that students may have difficulties in figuring out how to take away “b” *before* they have “b”. We also consider the inadequate understanding of the basic operations, e.g. the misinterpretation of $a - b + b$ as $a - (b + b)$. Students’ responses in follow-up interviews provide some evidence for these interpretations, as we shall see in the next paragraph.

Item (Questions)	Measure	Remarks
Q14: ■+■–○ = ( )	54.43	the most difficult question
Q12: ○–◇+◇ = ( )	53.67	↓
Q13: △+□–( ) = □	32.36	
Q11: □+○– ( ) = □	23.35	the least difficult question

Table 2: The item measures of Questions 11 to 14 for all students

To investigate further, we reviewed students’ explanations put forward in the interview. We found that the students always used such descriptions as “when you add the amount and then subtract the same amount, it means that nothing would change ...” to explain their action on transforming the inverse related problems. More often than not, their explanations went in one direction. They always explained in a way that if they first got a number and then it was reduced by itself, then nothing would have changed. They appeared to be more familiar with this kind of inverse related situations than those with the subtraction coming first before they got any amount for negation as in $a - b + b$. In Question 2 with 378 – 239 + 239, 3 out of the 18 interviewees disagreed that the “– 239” and “+ 239” can be set off. It is because when they faced the part of “– 239 + 239”, they only focused on the “239 + 239” and so they claimed that the two should not be set off by using the inverse principle. This indicates that some of the students are sometimes confused with the meaning of the operations, and thus cannot apply the inverse principle properly. Nevertheless, misunderstanding is not the only reason. The other 3 interviewees, who had answered Question 12 incorrectly in the written test but answered it correctly in the interview, claimed that they did not know why they had got it wrong at the very beginning or that they had not read it carefully before. We understand that some students may know the concept but they are not familiar with the presentation of the inverse related problems or they are just careless.

4. The students were affected by their general approach to computation problems

In general, as shown by the performances across Grades 2 to 4 in our results, students' ability of recognizing the inverse related problems is raised when the students are in a higher grade. However, the item measure of recognizing the five-term non-transparent inverse problems (i.e. Question 9 and 10) revealed a little discrepancy. In Grade 2, 12.9% of students are expected to identify the inverse related element in Questions 9 and 10. Meanwhile, the corresponding figure in Grade 4 is raised to 15.63% but that in Grade 3 drastically drops to 1.61% which is far below that in Grade 2.

The phenomenon that Grade 3 students performed extremely badly on identifying these five-term problems can be explained with reference to student responses in the interviews. One of the Grade 2 interviewees said, "Questions 9 and 10 were the most difficult questions because of the length of the questions. But if you have read it carefully, you can find the trick that the total of the third term and fourth term can be set off by the second term." On the other hand, most of the Grade 3 interviewees disagreed that Questions 9 and 10 were particularly difficult. They said, "Although there are more terms, the calculation process is the same." Although their comments cannot represent the whole group, not even students in general, their explanations show an obvious difference in students' general approaches to computational problems. For Grade 2 students, they may spend more time to analyse the five-term problems which appear to be complicated and unfamiliar to them. The analysis may lead the Grade 2 students to have a higher possibility of identifying the inverse related situation. But, for Grade 3 students (at least those in Hong Kong), they got more used to complicated numerical computation such as the present five-term calculation. Instead of the "careful reading" among Grade 2 students facing unfamiliar problems, Grade 3 students just do the calculation right away but not analysing the situation. Certainly, there are many other factors affecting students' performance in identifying the five-term non-transparent inverse problems. Otherwise, Grade 4 students should have shown the worst performance due to their familiarity with routine computation. However, we suggest that the general approach with which students adopt in handling computational problems is one of the factors for further investigation.

Conclusion and implication

The results show that the students understand the concept of inverse principle and are able to apply it in certain contexts – though better in some than in others. They do not use it as frequently and properly as desired, even in the cases of transparent inverse problems. Students do it better when they are in higher grades. But, as discussed above, caution has to be taken regarding the mechanical response of students to computational problems. While most of the students prefer using column arithmetic (as their routine expertise) in handling computational problems, they are weak in identifying the conceptual relation embedded in a situation and thus, notwithstanding the conceptual knowledge about the inverse principle, they cannot apply it to a possible shortcut for more effective calculation. Some of the students even consider column arithmetic as more accurate than a shortcut well grounded in a theoretical principle. When students are gaining computational competence, they should also develop their conceptual sophistication. This will, in the long run, enable students to analyse a problem in terms of structures and relations, and look for more suitable strategies and arithmetic principles to solve the problem more effectively. More importantly, we should note that structures and relations are crucial to algebra learning at secondary levels. Students must gradually progress from arithmetic computation with numbers to algebraic operations in abstract terms.

Moreover, we have noticed the effect brought about by the order of operations on students' identification of the inverse related element in computational problems. As far as we know, this effect has been noted, but only briefly, in previous research studies (cf. van den Heuvel-Panhuizen & Treffers, 2009, p. 105). We, from the perspective of a classroom teacher, consider that this order of operations is important for at least two reasons. One is concerned with the decision on setting teaching examples and test or exercise items, whereas the other is with the interpretation of students' weaknesses in handling subtraction or difference as compared with addition or sum.

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