On Thinking and Revising a Proof of an IMO Problem

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Article [1] introduced a simple proof of the following problem and its generalization. But *I am sorry* that there were mistakes in the proof. In this article, I will introduce another proof of this problem and its related results.

PROBLEM (the third problem of the 46 IMO) Let x, y, z be positive real numbers and $xyz \ge 1$. Prove that

$$\frac{x^5-x^2}{x^5+y^2+z^2}+\frac{y^5-y^2}{x^2+y^5+z^2}+\frac{z^5-z^2}{x^2+y^2+z^5}\ge 0 .$$

PROOF First, $\frac{x^5-x^2}{x^5+y^2+z^2}+\frac{y^5-y^2}{x^2+y^5+z^2}+\frac{z^5-z^2}{x^2+y^2+z^5}\ge 0 \iff$

$$\frac{x^5}{x^5+y^2+z^2}+\frac{y^5}{x^2+y^5+z^2}+\frac{z^5}{x^2+y^2+z^5}\ge\frac{x^2}{x^5+y^2+z^2}+\frac{y^2}{x^2+y^5+z^2}+\frac{z^2}{x^2+y^2+z^5}$$

Since x, y, z are positive real numbers and $xyz \ge 1$, we have

$$y^4+z^4-y^3z-yz^3=(y-z)^2(y^2+yz+z^2)\ge 0$$

and so

$$y^4+z^4\ge y^3z+yz^3=yz(y^2+z^2) .$$

Since $y^2+z^2\le xyz(y^2+z^2)\le x(y^4+z^4)$, we have

$$\frac{x^5}{x^5+y^2+z^2}\ge\frac{x^5}{x^5+x(y^4+z^4)}=\frac{x^4}{x^4+y^4+z^4} .$$

Similarly, $\frac{y^5}{x^2+y^5+z^2}\ge\frac{y^4}{x^4+y^4+z^4}$, $\frac{z^5}{x^2+y^2+z^5}\ge\frac{z^4}{x^4+y^4+z^4}$.

Summing up, we have $\frac{x^5}{x^5+y^2+z^2}+\frac{y^5}{x^2+y^5+z^2}+\frac{z^5}{x^2+y^2+z^5}\ge 1$.

Since $xyz\ge 1$, we have $(x^5+y^2+z^2)(yz+y^2+z^2)$

$$\ge (x^5+y^2+z^2)(\frac{1}{x}+y^2+z^2)=x^4+(x^5+\frac{1}{x})(y^2+z^2)+(y^2+z^2)^2$$

$\geq \quad x^4 + 2x^2(y^2+z^2) + (y^2+z^2)^2 = (x^2+y^2+z^2)^2$,

i.e. $\dfrac{1}{(x^5+y^2+z^2)(yz+y^2+z^2)} \leq \dfrac{1}{(x^2+y^2+z^2)^2}$

So, $\dfrac{x^2}{x^5+y^2+z^2} \leq \dfrac{x^2(yz+y^2+z^2)}{(x^2+y^2+z^2)^2} \leq \dfrac{x^2(\frac{y^2+z^2}{2}+y^2+z^2)}{(x^2+y^2+z^2)^2}$

$\leq \quad \dfrac{3x^2(y^2+z^2)}{2(x^2+y^2+z^2)^2}$.

So, $\dfrac{y^2}{x^2+y^5+z^2} \geq \dfrac{3y^2(z^2+x^2)}{2(x^2+y^2+z^2)^2}$, $\dfrac{z^2}{x^2+y^2+z^5} \geq \dfrac{3z^2(x^2+y^2)}{2(x^2+y^2+z^2)^2}$.

Hence, $\dfrac{x^2}{x^5+y^2+z^2} + \dfrac{y^2}{x^2+y^5+z^2} + \dfrac{z^2}{x^2+y^2+z^5} \leq \dfrac{3(x^2y^2+y^2z^2+z^2x^2)}{(x^2+y^2+z^2)^2}$.

Since $(x^2+y^2+z^2)^2 = x^4+y^4+z^4+2(x^2y^2+y^2z^2+z^2x^2)$

$\geq \quad x^2y^2+y^2z^2+z^2x^2+2(x^2y^2+y^2z^2+z^2x^2) = 3(x^2y^2+y^2z^2+z^2x^2)$,

we have $\dfrac{3(x^2y^2+y^2z^2+z^2x^2)}{(x^2+y^2+z^2)^2} \leq 1$. Thus

$$\frac{x^5}{x^5+y^2+z^2} + \frac{y^5}{x^2+y^5+z^2} + \frac{z^5}{x^2+y^2+z^5} \geq 1 \geq \frac{x^2}{x^5+y^2+z^2} + \frac{y^2}{x^2+y^5+z^2} + \frac{z^2}{x^2+y^2+z^5}$$

The above inequality is true.

By the method of the above proof, we have the following theorem.

THEOREM Let $x_1, x_2, \ldots, x_n$ be positive real numbers and $x_1x_2 \cdots x_n \geq 1$ where $n \geq 3$ and $n \in \mathbf{N}$. Then

$$\frac{x_1^{n+2}}{x_1^{n+2}+x_2^{2}+\ldots+x_n^{2}} + \frac{x_2^{n+2}}{x_1^{2}+x_2^{n+2}+\ldots+x_n^{2}} + \ldots + \frac{x_n^{n+2}}{x_1^{2}+x_2^{2}+\ldots+x_n^{n+2}} \geq 1 .$$

PROOF $\because$ $\left(\dfrac{x_1^{\alpha}+x_2^{\alpha}+\ldots+x_n^{\alpha}}{n}\right)^{\frac{1}{\alpha}} \geq \left(\dfrac{x_1^{\beta}+x_2^{\beta}+\ldots+x_n^{\beta}}{n}\right)^{\frac{1}{\beta}}$

where $\alpha, \beta \in \mathbf{N}^*$ and $\alpha \geq \beta$,

$\therefore \quad x_2^{n+1} + x_3^{n+1} + \ldots + x_n^{n+1} - x_2x_3 \cdots x_n(x_2^2 + x_3^2 + \ldots + x_n^2)$

$$\geq \quad (n-1)\left(\frac{x_2^{\ 2}+x_3^{\ 2}+\ldots+x_n^{\ 2}}{n-1}\right)^{\frac{n+1}{2}} - x_2x_3 \cdots x_n(x_2^2 + x_3^2 + \ldots + x_n^2)$$

$$= \quad (x_2^2 + x_3^2 + \ldots + x_n^2)\left(\frac{x_2^{\ 2}+x_3^{\ 2}+\ldots+x_n^{\ 2}}{n-1}\right)^{\frac{n-1}{2}} - x_2x_3 \cdots x_n(x_2^2 + x_3^2 + \ldots + x_n^2)$$

$$= \quad (x_2^2 + x_3^2 + \ldots + x_n^2)[\left(\frac{x_2^{\ 2}+x_3^{\ 2}+\ldots+x_n^{\ 2}}{n-1}\right)^{\frac{n-1}{2}} - x_2x_3 \cdots x_n]$$

$$\geq \quad (x_2^2 + x_3^2 + \ldots + x_n^2)[(x_2x_3\cdots x_n)^{\frac{2}{n-1}\times\frac{n-1}{2}} - x_2x_3 \cdots x_n] \quad = \quad 0 \ ,$$

i.e. $x_2^{n+1} + x_3^{n+1} + \ldots + x_n^{n+1} \geq x_2x_3 \cdots x_n(x_2^2 + x_3^2 + \ldots + x_n^2)$.

Since $x_2^2 + x_3^2 + \ldots + x_n^2 \leq x_1x_2 \cdots x_n(x_2^2 + x_3^2 + \ldots + x_n^2)$,
we have $x_2^2 + x_3^2 + \ldots + x_n^2 \leq x_1(x_2^{n+1} + x_3^{n+1} + \ldots + x_n^{n+1})$,

i.e. $$\frac{x_1^{\ n+2}}{x_1^{\ n+2}+x_2^{\ 2}+\ldots+x_n^{\ 2}} \geq \frac{x_1^{\ n+2}}{x_1^{\ n+2}+x_1(x_2^{\ n+1}+\ldots+x_n^{\ n+1})} = \frac{x_1^{\ n+1}}{x_1^{\ n+1}+x_2^{\ n+1}+\ldots+x_n^{\ n+1}}$$

Similarly, we have $\dfrac{x_2^{\ n+2}}{x_1^{\ 2}+x_2^{\ n+2}+\ldots+x_n^{\ 2}} \geq \dfrac{x_2^{\ n+1}}{x_1^{\ n+1}+x_2^{\ n+1}+\ldots+x_n^{\ n+1}}$, … ,

$\dfrac{x_n^{\ n+2}}{x_1^{\ 2}+x_2^{\ 2}+\ldots+x_n^{\ n+2}} \geq \dfrac{x_n^{\ n+1}}{x_1^{\ n+1}+x_2^{\ n+1}+\ldots+x_n^{\ n+1}}$. Summing up, we have

$$\frac{x_1^{\ n+2}}{x_1^{\ n+2}+x_2^{\ 2}+\ldots+x_n^{\ 2}} + \frac{x_2^{\ n+2}}{x_1^{\ 2}+x_2^{\ n+2}+\ldots+x_n^{\ 2}} + \ldots + \frac{x_n^{\ n+2}}{x_1^{\ 2}+x_2^{\ 2}+\ldots+x_n^{\ n+2}} \geq 1 \ .$$

Finally, I introduce the following conjecture.

CONJECTURE Let $x_1, x_2, \ldots, x_n$ be positive real numbers and $x_1x_2 \cdots x_n \geq 1$ where $n \geq 3$ and $n \in \mathbf{N}$. Then

$$\frac{x_1^{\ 2}}{x_1^{\ n+2}+x_2^{\ 2}+\ldots+x_n^{\ 2}} + \frac{x_2^{\ 2}}{x_1^{\ 2}+x_2^{\ n+2}+\ldots+x_n^{\ 2}} + \ldots + \frac{x_n^{\ 2}}{x_1^{\ 2}+x_2^{\ 2}+\ldots+x_n^{\ n+2}} \leq 1 \ .$$

Reference

[1] Zhang Yun (2006). A Simple Proof and Generalization of the Third Problem of the 46 IMO. *EduMath*, *22*, 105 – 107.

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