Apollonius Circle

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In Form Six Pure Mathematics, the following equations of loci are often encountered:

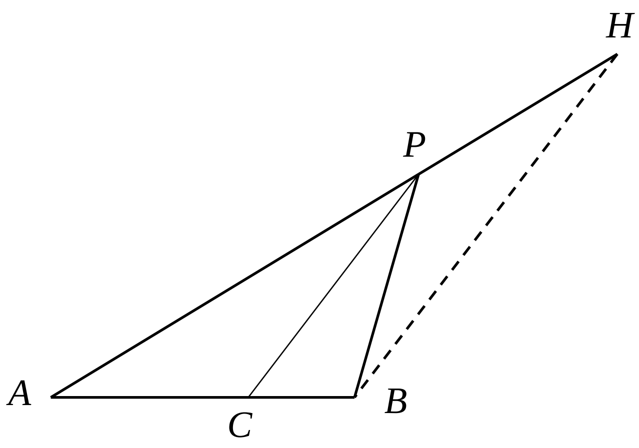
1. $|z-a|=r$;
2. $|z-a|=|z-b|$;
3. $|z-a|=k|z-b|$ where $k>0$ and $k\neq 1$.

The geometry of the first two equations is obvious. The first equation represents a circle with centre at a and radius r while the second represents the perpendicular bisector of a and b . The third equation can be shown to represent a circle. This result can be obtained algebraically by letting $z = x + yi$ and then simplifying. The third type of equation has appeared several times in the Advanced Level Pure Mathematics Examination (1998IQ4, 2002IQ2, 2008IQ5). However, the geometry behind the formation of a circle is not evident and rarely discussed in textbooks.

In fact, for two fixed points A and B in the plane, if $\frac{PA}{PB}=k$ (a constant) where $k>0$ and $k\neq 1$, the locus of P is called an *Apollonius Circle* (see reference 1). A brief explanation to this can be found in Gow (reference 2). The authors would like to elaborate the formation of this circle in details.

THEOREM 1 Let C be the internal point of division on AB such that $\frac{PA}{PB}=\frac{CA}{CB}=k$. Then PC is the angle bisector of $\angle APB$.

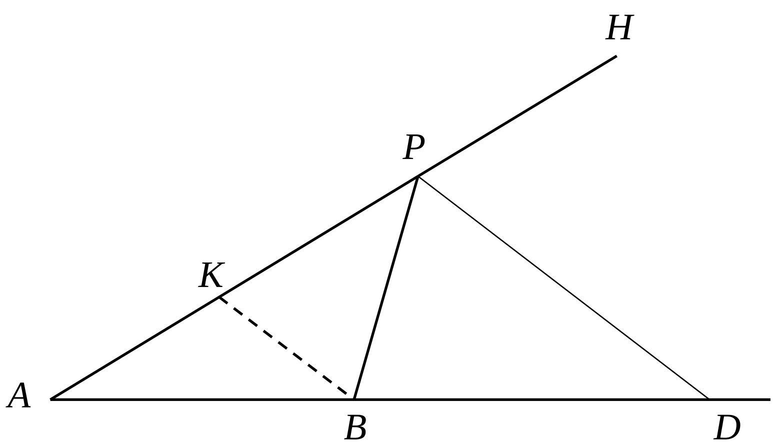
PROOF Through B, construct a line parallel to CP. This line meets AP produced at H.



$$\frac{AP}{PH} = \frac{AC}{CB} = \frac{PA}{PB}.$$ Therefore $PH = PB$.

	$\angle APC = \angle PHB$	(corr. $\angle$s, $CP \parallel BH$)
	$\angle CPB = \angle PBH$	(alt. $\angle$s, $CP \parallel BH$)
$\because$	$\angle PHB = \angle PBH$	(base $\angle$s of isos. Δ)
$\therefore$	$\angle APC = \angle CPB$	

THEOREM 2 Let D be the external point of division on AB such that $\frac{PA}{PB} = \frac{DA}{DB} = k$. Then PD is the angle bisector of $\angle BPH$ where H is a point on AP produced (as shown in the following figure).

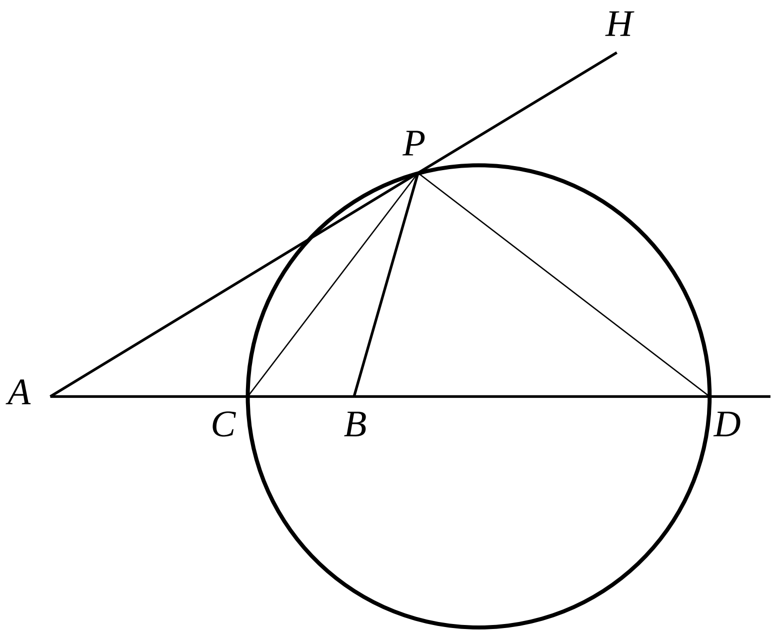


PROOF The proof is similar to that in THEOREM 1. Construct $BK \parallel DP$.

$\frac{AP}{KP} = \frac{AD}{BD} = \frac{PA}{PB}$. Therefore $PK = PB$.

	$\angle HPD = \angle PKB$	(corr. ∠s, $BK \parallel DP$)
	$\angle BPD = \angle PBK$	(alt. ∠s, $BK \parallel DP$)
∵	$\angle PKB = \angle PBK$	(base ∠s of isos. Δ)
∴	$\angle HPD = \angle BPD$	

Note that in THEOREM 1 and THEOREM 2, the internal point of division C and the external point of division D depend solely on A and B.



Hence, $\angle CPD + \angle BPD = 90°$ and P lies on a circle with CD as a diameter.

References

1. http://mathworld.wolfram.com/ApolloniusCircle.html
2. Margaret Gow (1960). *A Course in Pure Mathematics*. Butterworth-Heinemann, p.115.

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