A Property of Conic Section

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In this article, I will introduce a property of conic section.

THEOREM 1 Let a focus of an ellipse be F and two points on the ellipse be A and B such that A, B and F are collinear. Two tangents of the ellipse passing through A and B are drawn. If the point of intersection of the two tangents is M, then $FM \perp AB$.

PROOF Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ($a > b > 0$). Let $F(-c, 0)$ be its left focus. According to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we can get $\frac{2x}{a^2}+\frac{2yy'}{b^2}=1$ and so $y' = -\frac{b^2x}{a^2y}$.

Let the coordinates of A and B be (x_1, y_1) and (x_2, y_2) respectively. Let k_A and k_B be the slopes of the tangents at A and B respectively. Then $k_A = -\frac{b^2x_1}{a^2y_1}$, $k_B = -\frac{b^2x_2}{a^2y_2}$.

Let $\overrightarrow{AF} = \lambda\overrightarrow{FB}$ ($\lambda > 0$). Since $\overrightarrow{AF} = (-c - x_1, -y_1)$, $\overrightarrow{FB} = (x_2 + c, y_2)$, $-c - x_1 = \lambda x_2 + \lambda c$, $-y_1 = \lambda y_2$.

So $y_1 = -\lambda y_2$, $x_1 = -\lambda c - c - \lambda x_2$. (1)

According to $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1$ and (1), we have

$$\frac{(-\lambda c - c - \lambda x_2)^2}{a^2}+\frac{(-\lambda y_2)^2}{b^2}=1$$ and hence

$$b^2\lambda^2c^2 + b^2c^2 + b^2\lambda^2x_2^2 + 2b^2c^2\lambda + 2b^2c\lambda^2x_2 + 2b^2c\lambda x_2 + a^2\lambda^2y_2^2 = a^2b^2 . \qquad (2)$$

According to $\frac{x_2^{\ 2}}{a^2}+\frac{y_2^{\ 2}}{b^2}=1$, we have

$$b^2\lambda^2 x_2^{\ 2} + a^2\lambda^2 y_2^{\ 2} = a^2 b^2 \lambda^2 . \qquad (3)$$

(2) – (3) : $$2b^2 c\lambda(1+\lambda)x_2 = a^2b^2 - a^2b^2\lambda^2 - b^2c^2\lambda^2 - 2b^2c^2\lambda - b^2c^2$$

$$2c\lambda(1+\lambda)x_2 = a^2(1-\lambda^2) - c^2(1+\lambda)^2$$

$$2c\lambda x_2 = a^2(1-\lambda) - c^2(1+\lambda)$$

$$= b^2 - \lambda(a^2+c^2) , \quad \text{as} \quad a^2 - c^2 = b^2 .$$

$$x_2 = \frac{b^2-\lambda(a^2+c^2)}{2c\lambda}$$

Hence $x_1 = -\lambda c - c - \lambda\cdot\frac{b^2-\lambda(a^2+c^2)}{2c\lambda} = \frac{b^2\lambda-2c^2-b^2}{2c}$.

The equation of tangent at A is

$$y - y_1 = -\frac{b^2x_1}{a^2y_1}(x-x_1) = -\frac{b^2x_1}{a^2y_1}x+\frac{b^2x_1^{\ 2}}{a^2y_1} .$$

So $y = -\frac{b^2x_1}{a^2y_1}x+\frac{b^2}{y_1}$. (4)

Similarly, the equation of tangent at B is $y = -\frac{b^2x_2}{a^2y_2}x+\frac{b^2}{y_2}$. (5)

By (4) and (5), we have $x = \frac{a^2(y_1-y_2)}{x_2y_1-x_1y_2}$ (6)

Combining (1) and (6), we can get $x = -\frac{a^2}{c}$.

Combining $x = -\frac{a^2}{c}$ and (4), we have

$$y = \frac{b^2(c+x_1)}{cy_1} = \frac{b^2c+b^2x_1}{cy_1} .$$

Thus $M = \left(-\dfrac{a^2}{c}\ ,\ \dfrac{b^2c + b^2x_1}{cy_1}\right)$.

$\overrightarrow{FM} = \left(-\dfrac{a^2}{c} + c\ ,\ \dfrac{b^2c + b^2x_1}{cy_1}\right) = \left(-\dfrac{b^2}{c}\ ,\ \dfrac{b^2c + b^2x_1}{cy_1}\right)$,

$\overrightarrow{AB} = (x_2 - x_1\ , y_2 - y_1) = (x_2 + \lambda c + c + \lambda x_2\ , y_2 + \lambda y_2) = (1 + \lambda)(x_2 + c\ , y_2)$.

Then $\overrightarrow{FM} \cdot \overrightarrow{AB} = -\dfrac{b^2}{c}(1 + \lambda)(x_2 + c) + \dfrac{b^2 y_2}{cy_1}(1 + \lambda)(c + x_1)$

$= -\dfrac{b^2}{c}(1 + \lambda)(x_2 + c) - \dfrac{b^2}{c\lambda}(1 + \lambda)(c + x_1)$

$= -\dfrac{b^2}{c\lambda}(1 + \lambda)(\lambda x_2 + \lambda c + c + x_1) = -\dfrac{b^2}{c\lambda}(x_1 + \lambda x_2 + c + c\lambda)$.

By (1), we know $x_1 + \lambda x_2 + c + c\lambda = 0$. So $\overrightarrow{FM} \cdot \overrightarrow{AB} = 0$.

Therefore, $FM \perp AB$ and THEOREM 1 is proved.

THEOREM 2 Let a focus of a hyperbola be F and two points on the hyperbola be A and B such that A , B and F are collinear. Two tangents of the hyperbola passing through A and B are drawn. If the point of intersection of the two tangents is M , then $FM \perp AB$.

The proof of THEOREM 2 is similar to that of THEOREM 1. So it is omitted.

THEOREM 3 Let the focus of a parabola be F and two points on the parabola be A and B such that A , B and F are collinear. Two tangents of the parabola passing through A and B are drawn. If the point of intersection of the two tangents is M , then $FM \perp AB$.

PROOF Let the equation of the parabola be $x^2 = 2py$.

Then we have $y = \dfrac{x^2}{2p}$, focus $F\left(0\ , \dfrac{p}{2}\right)$ and $y' = \dfrac{x}{p}$.

We put $A(x_1\ , y_1)$, $B(x_2\ , y_2)$. Let k_A and k_B be the slopes of the tangents at A and B respectively. Then $k_A = \dfrac{x_1}{p}$, $k_B = \dfrac{x_2}{p}$.

The equation of tangent at A is $y - y_1 = \frac{x_1}{p}(x - x_1) = \frac{x_1}{p}x - \frac{x_1^2}{p}$

Since $y_1 = \frac{x_1^2}{2p}$, $y = \frac{x_1}{p}x - y_1$. (7)

Similarly, the equation of tangent at B is $y = \frac{x_2}{p}x - y_2$. (8)

By (7) and (8), we have $M = (\frac{x_1 + x_2}{2}, \frac{x_1 x_2}{2p})$.

Since A, B and F are collinear, there is a positive real number λ such that $\overrightarrow{AF} = \lambda\overrightarrow{FB}$.

As $\overrightarrow{AF} = (-x_1, \frac{p}{2} - y_1)$ and $\overrightarrow{FB} = (x_2, y_2 - \frac{p}{2})$, we have $-x_1 = \lambda x_2$ and $\frac{p}{2} - y_1 = \lambda(y_2 - \frac{p}{2})$.

So $x_1 = -\lambda x_2$ and $y_1 = \frac{p}{2} + \frac{p\lambda}{2} - \lambda y_2$.

Then $\frac{x_1^2}{2p} = \frac{p}{2} + \frac{p\lambda}{2} - \lambda \cdot \frac{x_2^2}{2p}$. $\frac{\lambda^2 x_2^2}{2p} = \frac{p}{2} + \frac{p\lambda}{2} - \frac{\lambda x_2^2}{2p}$.

Thus $x_2 = \pm\frac{p}{\sqrt{\lambda}}$. Taking $x_2 = \frac{p}{\sqrt{\lambda}}$, we have $x_1 = -p\sqrt{\lambda}$.

Hence $x_1 x_2 = -p^2$, $x_1 + x_2 = \frac{p - p\lambda}{\sqrt{\lambda}}$ and $M = (\frac{p - p\lambda}{2\sqrt{\lambda}}, -\frac{p}{2})$.

Since $\overrightarrow{FM} = (\frac{p - p\lambda}{2\sqrt{\lambda}}, -p)$, $\overrightarrow{AB} = (x_2 - x_1, y_2 - y_1) = (\frac{p + p\lambda}{\sqrt{\lambda}}, \frac{p(1-\lambda^2)}{2\lambda})$, and hence $\overrightarrow{FM} \cdot \overrightarrow{AB} = \frac{p^2(1-\lambda^2)}{2\lambda} - \frac{p^2(1-\lambda^2)}{2\lambda} = 0$.

So $FM \perp AB$. Similarly, we also have $FM \perp AB$ if $y^2 = 2px$. Therefore, THEOREM 3 is proved.

Combining THEOREM 1, THEOREM 2 and THEOREMK 3, we can get:

THEOREM 4 Let a focus of a conic section be F and two points on the conic section be A and B such that A, B and F are collinear. Two tangents of the conic section passing through A and B are drawn. If the point of intersection of the two tangents is M, then $FM \perp AB$. Furthermore, M is a point on the directrix of the conic section corresponding to the focus F.

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Editor's Note

Zhang Yun has stated and proved a very beautiful property in conic section which is not often mentioned in usual textbooks. One of our reviewers has found a shorter way to prove this property and it is given as follows:

Let $A(x_2, y_2)$ and $B(x_3, y_3)$ be two points lying on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and $M(x_1, y_1)$ be the intersection of the tangents at A and B.

Since the equation of the tangent at A is $\frac{xx_2}{a^2}+\frac{yy_2}{b^2}=1$ and this tangent passes through $M(x_1, y_1)$, we have $\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}=1$.

In other words, $A(x_2, y_2)$ satisfies the equation $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$.

By similar argument, $B(x_3, y_3)$ also satisfies the equation $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$.

As $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$ is linear, it must be the equation of AB.

Let $F(c, 0)$ be a focus of the ellipse. (Then $c^2 = a^2 - b^2$.)

If AB passes through F, we have $\frac{cx_1}{a^2}=1 \Rightarrow x_1 = \frac{a^2}{c}$.

Hence, M lies on a directirx of the ellipse.

Slope of $AB = -\frac{b^2x_1}{a^2y_1}=-\frac{b^2}{cy_1}$, from the equation of AB.

Slope of $FM = \frac{y_1}{x_1-c}=\frac{y_1}{\frac{a^2}{c}-c}=\frac{cy_1}{a^2-c^2}=\frac{cy_1}{b^2}$, as $c^2 = a^2 - b^2$.

$\because$ slope of AB $\times$ slope of $MF = -1$,

$\therefore$ $FM \perp AB$