Several Proofs of Ceva's Theorem by Students

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Introduction

In this article, some proofs of Ceva's Theorem by students are presented. The proofs were taken from the homework given to the students in a Geometry workshop held in the summer 2007.

Ceva's Theorem. Let P, Q and R be points on the side BC, CA and AB of triangle ABC respectively. If AP, BQ and CR are concurrent, then $\frac{AR}{RB}\cdot\frac{BP}{PC}\cdot\frac{CQ}{QA}=1$.

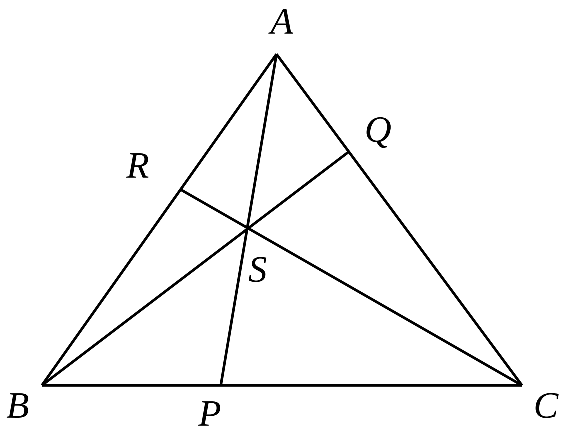

Figure 1

The proofs of Ceva's Theorem that appeared in most classical geometry books are similar to the proof below, which involved ratios of areas of triangles.

The Usual Proof of Ceva's Theorem

Let S be the point of concurrency (figure 1). For simplicity, let $[XYZ]$ denote the area of any triangle XYZ. Since the areas of triangles with same height are proportional to bases, we have $\frac{AR}{RB}=\frac{[ARS]}{[RBS]}=\frac{[ARC]}{[RBC]}$.

By property of proportions, $\frac{AR}{RB}=\frac{[ARC]-[ARS]}{[RBC]-[RBS]}=\frac{[ASC]}{[CSB]}$.

Similarly, $\dfrac{BP}{PC}=\dfrac{[BSA]}{[ASC]}$ and $\dfrac{CQ}{QA}=\dfrac{[CSB]}{[BSA]}$.

So, $\dfrac{AR}{RB}\cdot\dfrac{BP}{PC}\cdot\dfrac{CQ}{QA}=\dfrac{[ASC]}{[CSB]}\cdot\dfrac{[BSA]}{[ASC]}\cdot\dfrac{[CSB]}{[BSA]}=1$.

Six Different Proofs of Ceva's Theorem by the Students

Proof 1. (by Monzon Abel, Chan Chi Fai, Chan Ka Lun, Chan Wing Yan Joyce, Hui Chiu Kit, Hui Man Wa, Wendy Chow, Kan Ka Yin, Lam Ching Yan, Lam Hiu Yue, Leung Ka Hei, Lo Ching Hoi, Mok Tung Hoi, Ning Man Chit, So Tsz Chung, Tan Pak Chiu, Nicole Tin, Tsang Ting Ho, Wong Cheuk Kwan, Wong Ching, Wong See Wai, Yu Lik Hang and Anglea Wu)

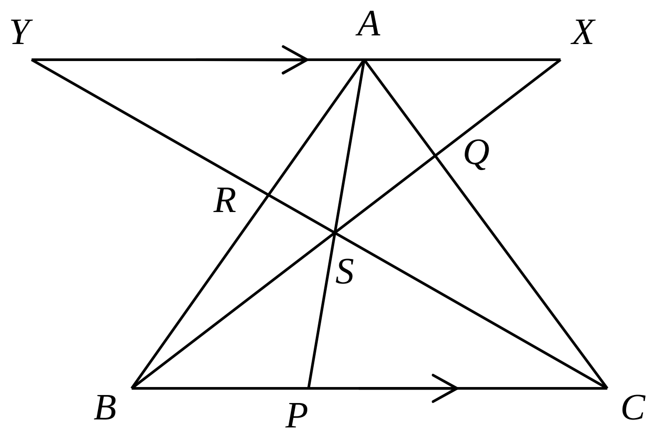

Figure 2

Let X and Y be points on BQ produced and CR produced respectively such that XAY is a straight line parallel to BC (figure 2).

As $\Delta ARY \sim \Delta BRC$ and $\Delta AQX \sim \Delta CQB$, we have

$$\frac{AR}{RB}=\frac{YA}{BC} \quad \text{and} \quad \frac{CQ}{QA}=\frac{BC}{AX} .$$

As $\Delta ASX \sim \Delta PSB$ and $\Delta ASY \sim \Delta PSC$, we have

$$\frac{BP}{AX}=\frac{PS}{SA}=\frac{PC}{YA} , \quad \text{and so} \quad \frac{BP}{PC}=\frac{AX}{YA} .$$

Hence $\dfrac{AR}{RB}\cdot\dfrac{BP}{PC}\cdot\dfrac{CQ}{QA}=\dfrac{YA}{BC}\cdot\dfrac{AX}{YA}\cdot\dfrac{BC}{AX}=1$.

Proof 2. (by Kwan Chin Wai, Tai Wai Ming)

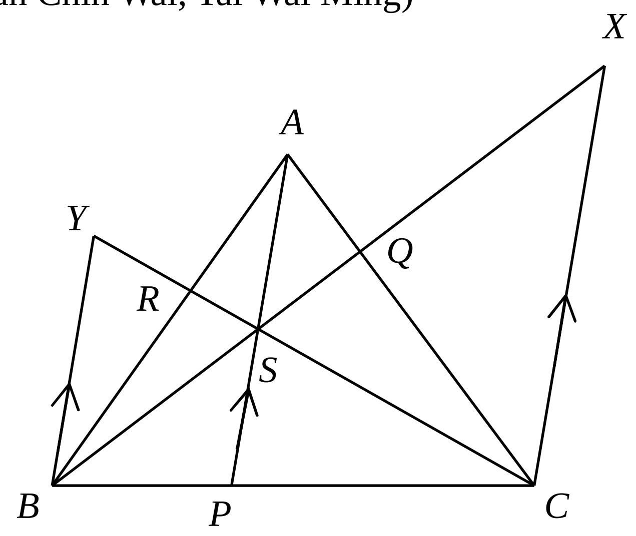

Figure 3

Let X and Y be points on BQ produced and CR produced respectively such that BY and CX are both parallel to PA (figure 3).

As $\Delta ARS \sim \Delta BRY$ and $\Delta AQS \sim \Delta CQX$, we have

$$\frac{AR}{RB} = \frac{SA}{BY} \quad \text{and} \quad \frac{CQ}{QA} = \frac{CX}{SA} \ .$$

As $AP \mathbin{//} XC$ and $\Delta BYS \sim \Delta XCS$, we have $\dfrac{BP}{PC} = \dfrac{BS}{SX} = \dfrac{BY}{CX}$.

Hence $\dfrac{AR}{RB} \cdot \dfrac{BP}{PC} \cdot \dfrac{CQ}{QA} = \dfrac{SA}{BY} \cdot \dfrac{BY}{CX} \cdot \dfrac{CX}{SA} = 1$.

Proof 3. (by Kan Ka Wing)

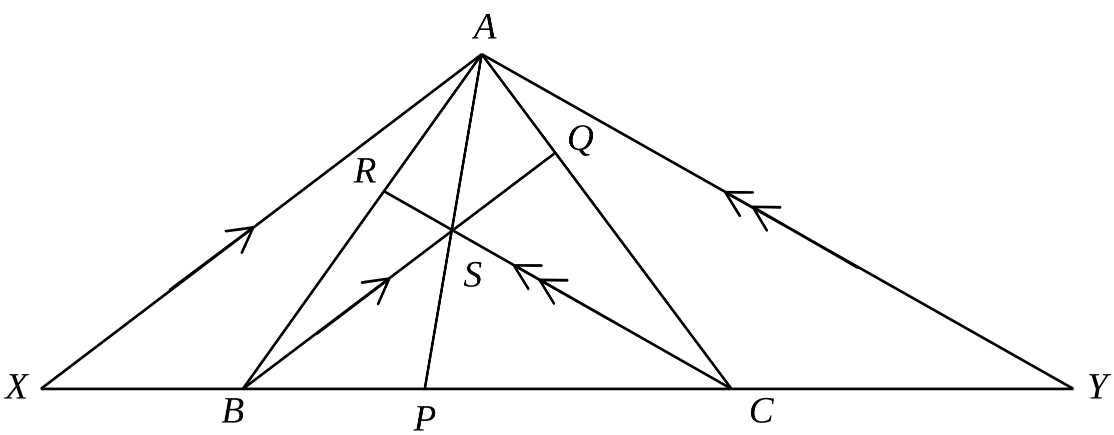

Figure 4

Let X and Y be points on the line BC such that $AX \mathbin{//} BQ$ and $AY \mathbin{//} RC$ (figure 4).

As $\Delta CQB \sim \Delta CAX$ and $\Delta BRC \sim \Delta BAY$, we have

$$\frac{CQ}{QA} = \frac{BC}{BX} \quad \text{and} \quad \frac{AR}{RB} = \frac{CY}{BC} \ .$$

As $\Delta BPS \sim \Delta XPA$ and $\Delta PSC \sim \Delta PAY$, we have

$$\frac{BP}{XB} = \frac{PS}{SA} = \frac{PC}{CY} \ , \quad \text{and so} \quad \frac{BP}{PC} = \frac{XB}{CY} \ .$$

Hence $\dfrac{AR}{RB} \cdot \dfrac{BP}{PC} \cdot \dfrac{CQ}{QA} = \dfrac{CY}{BC} \cdot \dfrac{BX}{CY} \cdot \dfrac{BC}{BX} = 1$.

Proof 4. (by Wong Wai Lung)

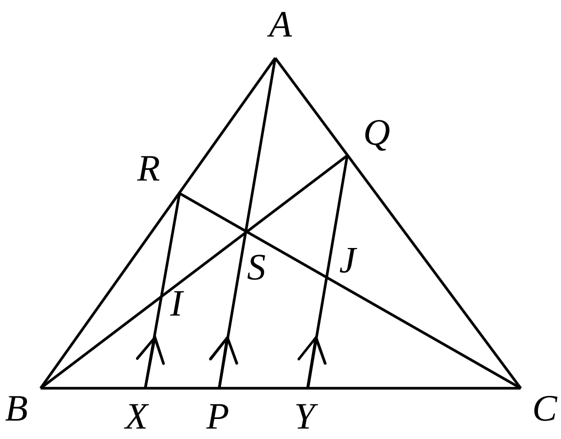

Figure 5

Let X and Y be points on BC such that XR and YQ are both parallel to PA. Let I be the point of intersection of BQ and XR, and J be the point of intersection of CR and YQ (figure 5).

By the intercept theorem, we have $\dfrac{AR}{RB} = \dfrac{PX}{XB}$ and $\dfrac{CQ}{QA} = \dfrac{CY}{YP}$.

As $XR \mathbin{//} PA$ and $\Delta BIR \sim \Delta BSA$, we have $\dfrac{PB}{XB} = \dfrac{SB}{IB} = \dfrac{AS}{RI}$.

As $YQ \mathbin{//} PA$ and $\Delta CJQ \sim \Delta CSA$, we have $\dfrac{CY}{CP} = \dfrac{CJ}{CS} = \dfrac{QJ}{AS}$.

Hence $\dfrac{AR}{RB}\cdot\dfrac{BP}{PC}\cdot\dfrac{CQ}{QA}=\dfrac{PX}{XB}\cdot\dfrac{BP}{PC}\cdot\dfrac{CY}{YP}=\dfrac{BP}{XB}\cdot\dfrac{CY}{PC}\cdot\dfrac{PX}{YP}$
$=\dfrac{AS}{RI}\cdot\dfrac{QJ}{AS}\cdot\dfrac{RS}{SJ}=\dfrac{QJ}{RI}\cdot\dfrac{RS}{SJ}=1$.

The last equality follows as $\Delta RIS \sim \Delta JQS$.

Proof 5. (by Hui Shun On)

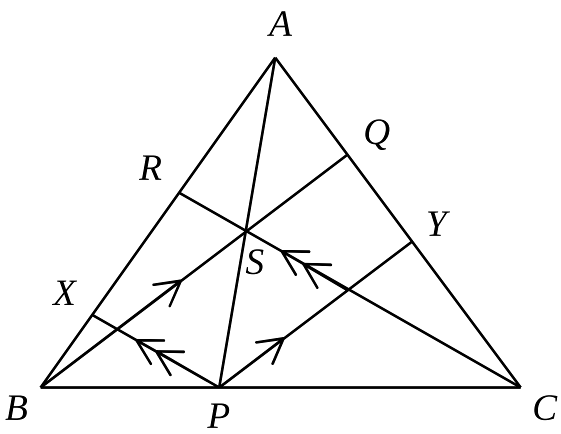

Figure 6

Let X be the point on AB such that PX is parallel to CR . Let Y be the point on AC such that PY is parallel to BQ (figure 6).

By the intercept theorem, we have

$$\frac{AR}{RX}=\frac{AS}{SP}=\frac{AQ}{QY}\ ,\quad \frac{RX}{RB}=\frac{PC}{BC}\quad \text{and}\quad \frac{CQ}{YQ}=\frac{BC}{BP}\ .$$

Hence $\dfrac{AR}{RB}\cdot\dfrac{BP}{PC}\cdot\dfrac{CQ}{QA}=\left(\dfrac{AR}{RX}\cdot\dfrac{RX}{RB}\right)\cdot\dfrac{BP}{PC}\cdot\left(\dfrac{YQ}{QA}\cdot\dfrac{CQ}{YQ}\right)$
$=\dfrac{AQ}{QY}\cdot\dfrac{PC}{BC}\cdot\dfrac{BP}{PC}\cdot\dfrac{YQ}{QA}\cdot\dfrac{BC}{BP}=1$.

Proof 6. (by Leung Yeung Yeung)

Let X and Y be points on AP produced such that BX is parallel to RC and CY is parallel to QB (figure 7).

By the intercept theorem, we have

$$\frac{AR}{RB}=\frac{AS}{SX}\quad \text{and}\quad \frac{CQ}{QA}=\frac{SY}{AS}\ .$$

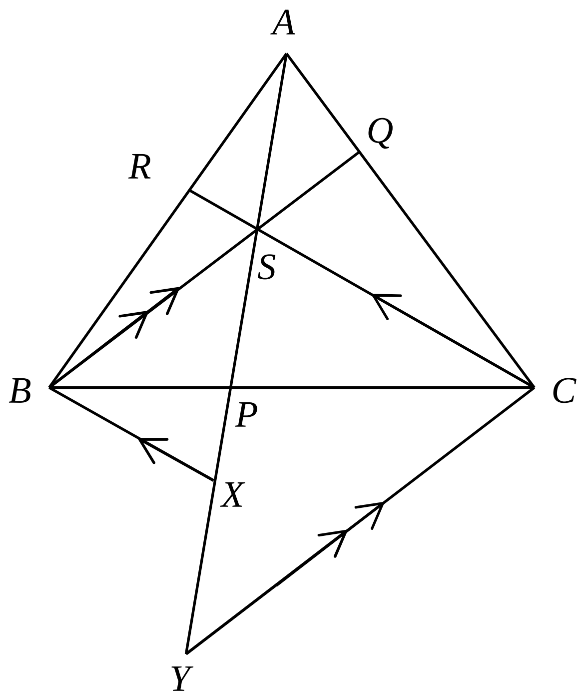

Figure 7

As $\Delta BPS \sim \Delta CPY$ and $\Delta BSX \sim \Delta CYS$, we have

$$\frac{BP}{PC} = \frac{SB}{CY} = \frac{SX}{SY} \quad .$$

Hence $$\frac{AR}{RB} \cdot \frac{BP}{PC} \cdot \frac{CQ}{QA} = \frac{AS}{SX} \cdot \frac{SX}{SY} \cdot \frac{SY}{AS} = 1 \quad .$$

Remarks

The "Usual Proof" is adopted from *Geometry Revisited* by Coxeter and Greitzer (1967). Some of the students' proofs appear also in some geometry textbooks. For example, proofs similar to "Proof 1" and "Proof 2" are also presented in Posamentier's book *Advanced Euclidean Geometry* (2002).

One common point of all the proofs by the students is the use of auxiliary parallel lines on the triangle to relate ratios of segments to other ratios of segments. During the workshop, I encouraged the students to first assume some numerical values on the ratios $AR : RB$ and $CQ : QA$ and try to calculate the remaining ratio $BP : PC$ with the help of adding auxiliary parallel lines. By this concrete computation, students were able to understand the relationships between ratios of the line segments on the triangle. The proofs then should be followed by generalizing $AR:RB$ and $CQ:QA$ to any arbitrary ratios.

The aim of this homework was to encourage students' participation in proving geometric theorems rather than memorizing proofs. I hoped that the students could develop some generic problem solving skills for future geometric problems.

Reference

Coxeter, H.S.M., Greitzer, S. L. (1967). *Geometry Revisited*. Mathematical Association of America.

Posamentier, Alfred S. (2002). *Advanced Euclidean Geometry*. Key College Publishing.

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