Some Theorems and Counter Examples on Differentiation

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In learning Advanced Level Pure Mathematics, some students are confused by the concept of differentiability of a function. In this short article, two examples and two theorems are presented, which will be helpful to teachers in teaching the concept of differentiability of a function.

Consider the following standard question:

Let $f(x) = x^{\frac{1}{3}}$ *. For* $x \neq 0$ *, find* $f'(x)$ *and prove that* $f'(0)$ *does not exist.*

For the first part, students can easily write down the correct answer $f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$. But in the second part, some students may give a proof like:

$\because \quad \lim_{x\to 0} \frac{1}{3}x^{-\frac{2}{3}} = \infty$,

$\therefore \quad f'(0)$ does not exist.

The above proof is not appropriate because the student only prove that $\lim_{x\to 0} f'(x)$ does not exist. This leads to the following questions: Is there any relation between $\lim_{x\to a} f'(x)$ and $f'(a)$? Can the existence of one imply the existence of the other? Furthermore, if both exist, are they necessarily the same?

First, let us consider the case when $\lim_{x\to a} f'(x)$ exists.

Counter Example 1 *There is a function* $f(x)$ *for which* $\lim_{x\to 0} f'(x)$ *exists but* $f'(0)$ *does not exist.*

Consider the function $f(x) = \begin{cases} x^2 & \text{, when } x \le 0 \\ x^2+1 & \text{, when } x > 0 \end{cases}$. It is not difficult to see that $\lim_{x\to 0^+} f'(x) = 0$ and $\lim_{x\to 0^-} f'(x) = 0$, thus $\lim_{x\to 0} f'(x) = 0$. However $f'(0)$ does not exist as this function is not continuous at $x = 0$.

But if $f(x)$ is continuous at $x = 0$ and $\lim_{x\to 0} f'(x)$ exists, are we sure that $f'(0)$ exists? The answer is yes, which is the following theorem, a question taken from the book by Apostol:

Theorem 1 *Let $f(x)$ be continuous on (a, b) with finite derivative $f'(x)$ everywhere in (a, b), except possibly at c. If $\lim_{x\to c} f'(x)$ exists and has the value A, then $f'(c)$ must also exist and have the value A.*

Proof For any $x \in (a, c)$, using the mean value theorem, we have a $\xi \in (x, c)$ such that $\frac{f(x)-f(c)}{x-c} = f'(\xi)$, so $\lim_{x\to c^-} \frac{f(x)-f(c)}{x-c} = A$, i.e. $f_-'(c) = A$.

Similarly, we can prove that $f_+'(c) = A$ and hence $f'(c) = A$.

Counter Example 2 *There is a function $f(x)$ for which $f'(0)$ exists but $\lim_{x\to 0} f'(x)$ does not exist, i.e. $f(x)$ is a differentiable function but $f'(x)$ is not continuous.*

Consider the function $f(x) = \begin{cases} x^{\frac{4}{3}} \sin\frac{1}{x} & \text{, when } x \ne 0 \\ 0 & \text{, when } x = 0 \end{cases}$.

From the first principles, we have $f'(0) = 0$.

But for $x \ne 0$, $f'(x) = \frac{4}{3} x^{\frac{1}{3}} \sin\frac{1}{x} - x^{-\frac{2}{3}} \cos\frac{1}{x}$, clearly $\lim_{x\to 0} f'(x)$ does not exist.

Lastly, let us end our discussion by giving the following result, which can be considered as a corollary of the theorem 1:

Theorem 2 *Assume that* $f(x)$ *has a finite derivative at each point of the open interval* (a, b). *Assume also that* $\lim_{x \to c} f'(x)$ *exists and is finite for some interior point* c. *Then the value of this limit must be* $f'(c)$.

From this theorem, we can see that if both $\lim_{x \to a} f'(x)$ and $f'(a)$ exist, they must be the same.

It is hope that the above counter examples and theorems can help the students to better understand the concepts of "differentiable" and "continuously differentiable".

Reference

Apostol Tom M. (1974). *Mathematical Analysis (2nd Ed.).* Addison-Wesley Publishing Company.

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