A Simple Proof and Generalization of the Third Problem of the 46 IMO

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The third problem of the 46 IMO is

Let x , y , z *be positive real numbers and* $xyz \geq 1$. *Prove that*

$$\frac{x^5-x^2}{x^5+y^2+z^2}+\frac{y^5-y^2}{x^2+y^5+z^2}+\frac{z^5-z^2}{x^2+y^2+z^5}\geq 0 \quad .$$

I will introduce a simple proof and generalization of this problem in this article.

Proof Since x , y , z are positive real numbers and $xyz \geq 1$,

$$\frac{x^5-x^2}{x^5+y^2+z^2}=\frac{x^2(x^3-1)}{x^5+y^2+z^2}\geq\frac{x^2(x^3-xyz)}{x^5+xy^3z+xyz^3}=\frac{x^2(x^2-yz)}{x^4+y^3z+yz^3} \quad .$$

Since $y^4+z^4-y^3z-yz^3=(y-z)^2(y^2+yz+z^2)\geq 0$,

$y^4+z^4\geq y^3z+yz^3$ and $\dfrac{x^2(x^2-yz)}{x^4+y^3z+yz^3}\geq\dfrac{x^2(x^2-yz)}{x^4+y^4+z^4}$. Similarly,

$$\frac{y^5-y^2}{y^5+z^2+x^2}\geq\frac{y^2(y^2-zx)}{x^4+y^4+z^4} \quad \text{and} \quad \frac{z^5-z^2}{z^5+x^2+y^2}\geq\frac{z^2(z^2-xy)}{x^4+y^4+z^4} \quad .$$

So, $\dfrac{x^5-x^2}{x^5+y^2+z^2}+\dfrac{y^5-y^2}{x^2+y^5+z^2}+\dfrac{z^5-z^2}{x^2+y^2+z^5}\geq\dfrac{x^2(x^2-yz)}{x^4+y^4+z^4}+$

$$\frac{y^2(y^2-zx)}{x^4+y^4+z^4}+\frac{z^2(z^2-xy)}{x^4+y^4+z^4}=\frac{x^4+y^4+z^4-xyz(x+y+z)}{x^4+y^4+z^4} \quad .$$

Since $\left(\dfrac{x^4+y^4+z^4}{3}\right)^{\frac{1}{4}}\geq\dfrac{x+y+z}{3}\geq\sqrt[3]{xyz}$,

$$x^4+y^4+z^4-xyz(x+y+z)\geq\frac{(x+y+z)^4}{27}-xyz(x+y+z)$$

$= (x + y + z)[\left(\dfrac{x+y+z}{3}\right)^3 - xyz] \geq 0$. So $\dfrac{x^4+y^4+z^4-xyz(x+y+z)}{x^4+y^4+z^4} \geq 0$. Hence $\dfrac{x^5-x^2}{x^5+y^2+z^2}+\dfrac{y^5-y^2}{x^2+y^5+z^2}+\dfrac{z^5-z^2}{x^2+y^2+z^5} \geq 0$.

Generally, we have

Theorem Let $x_1, x_2, \ldots, x_n$ be positive real numbers and $x_1x_2 \cdots x_n \geq 1$, $n \geq 3$ and $n \in \mathbf{N}^+$. Then $\dfrac{x_1^{n+2}-x_1^2}{x_1^{n+2}+x_2^2+\ldots+x_n^2}$ + $\dfrac{x_2^{n+2}-x_2^2}{x_2^{n+2}+x_3^2+\ldots+x_1^2}$ + … + $\dfrac{x_n^{n+2}-x_n^2}{x_n^{n+2}+x_1^2+\ldots+x_{n-1}^2} \geq 0$.

Proof Since $x_1, x_2, \ldots, x_n$ are positive real numbers and $x_1x_2 \cdots x_n \geq 1$,

$$\frac{x_1^{n+2}-x_1^2}{x_1^{n+2}+x_2^2+\ldots+x_n^2} = \frac{x_1^2(x_1^n-1)}{x_1^{n+2}+x_2^2+\ldots+x_n^2}$$

$$\geq \frac{x_1^2(x_1^n-x_1x_2\cdots x_n)}{x_1^{n+2}+x_1x_2^3x_3\cdots x_n+x_1x_2x_3^3x_4\cdots x_n+\ldots+x_1x_2\cdots x_{n-1}x_n^3}$$

$$= \frac{x_1^2(x_1^{n-1}-x_2\cdots x_n)}{x_1^{n+1}+x_2^3x_3\cdots x_n+x_2x_3^3x_4\cdots x_n+\ldots+x_2\cdots x_{n-1}x_n^3} .$$

Since $x_2^{n+1} + x_3^{n+1} + \ldots + x_n^{n+1} - x_2^3x_3 \cdots x_n - x_2x_3^3x_4 \cdots x_n - \ldots - x_2x_3 \cdots x_{n-1}x_n^3 = x_2^{n+1} + x_3^{n+1} + \ldots + x_n^{n+1} - x_2x_3 \cdots x_n(x_2^2 + x_3^2 + \ldots + x_n^2)$ and

$$\left(\frac{x_1^\alpha+x_2^\alpha+\ldots+x_n^\alpha}{n}\right)^{\frac{1}{\alpha}} \geq \left(\frac{x_1^\beta+x_2^\beta+\ldots+x_n^\beta}{n}\right)^{\frac{1}{\beta}} \quad (\alpha, \beta \in \mathbf{N}^+, \alpha \geq \beta) ,$$

$$x_2^{n+1} + x_3^{n+1} + \ldots + x_n^{n+1} - x_2x_3 \cdots x_n(x_2^2 + x_3^2 + \ldots + x_n^2)$$

$$\geq (n-1)\left(\frac{x_2^2+x_3^2+\ldots+x_n^2}{n-1}\right)^{\frac{n+1}{2}} - x_2x_3 \cdots x_n(x_2^2 + x_3^2 + \ldots + x_n^2)$$

$$= (x_2^2 + x_3^2 + \ldots + x_n^2) \times \left(\frac{x_2^2+x_3^2+\ldots+x_n^2}{n-1}\right)^{\frac{n-1}{2}} - x_2x_3 \cdots x_n(x_2^2 + x_3^2 + \ldots + x_n^2)$$

$$\geq (x_2^2 + x_3^2 + \ldots + x_n^2)\left[\left(\frac{x_2^2 + x_3^2 + \ldots + x_n^2}{n-1}\right)^{\frac{n-1}{2}} - x_2 x_3 \cdots x_n\right]$$

$$\geq (x_2^2 + x_3^2 + \ldots + x_n^2)\left[(x_2 x_3 \cdots x_n)^{\frac{2}{n-1}\times\frac{n-1}{2}} - x_2 x_3 \cdots x_n\right] = 0 .$$

So $\dfrac{x_1^{n+2} - x_1^2}{x_1^{n+2} + x_2^2 + \ldots + x_n^2}$

$$\geq \frac{x_1^2(x_1^{n-1} - x_2 \cdots x_n)}{x_1^{n+1} + x_2^3 x_3 \cdots x_n + x_2 x_3^3 x_4 \cdots x_n + \ldots + x_2 \cdots x_{n-1} x_n^3}$$

$$\geq \frac{x_1^2(x_1^{n-1} - x_2 \cdots x_n)}{x_1^{n+1} + x_2^{n+1} + \ldots + x_n^{n+1}}$$

Similarly, $\dfrac{x_2^{n+2} - x_2^2}{x_2^{n+2} + x_3^2 + \ldots + x_1^2} \geq \dfrac{x_2^2(x_2^{n-1} - x_3 x_4 \cdots x_1)}{x_1^{n+1} + x_2^{n+1} + \ldots + x_n^{n+1}}$, … ,

$\dfrac{x_n^{n+2} - x_n^2}{x_n^{n+2} + x_1^2 + \ldots + x_{n-1}^2} \geq \dfrac{x_1^2(x_1^{n-1} - x_1 x_2 \cdots x_{n-1})}{x_1^{n+1} + x_2^{n+1} + \ldots + x_n^{n+1}}$. So

$$\frac{x_1^{n+2} - x_1^2}{x_1^{n+2} + x_2^2 + \ldots + x_n^2} + \frac{x_2^{n+2} - x_2^2}{x_2^{n+2} + x_3^2 + \ldots + x_1^2} + \ldots + \frac{x_n^{n+2} - x_n^2}{x_n^{n+2} + x_1^2 + \ldots + x_{n-1}^2}$$

$$\geq \frac{x_1^{n+1} + x_2^{n+1} + \ldots + x_n^{n+1} - x_1 x_2 \cdots x_n (x_1 + x_2 + \ldots + x_n)}{x_1^{n+1} + x_2^{n+1} + \ldots + x_n^{n+1}} .$$

Since $x_1^{n+1} + x_2^{n+1} + \ldots + x_n^{n+1} - x_1 x_2 \cdots x_n (x_1 + x_2 + \ldots + x_n)$

$$\geq n\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)^{n+1} - x_1 x_2 \cdots x_n (x_1 + x_2 + \ldots + x_n)$$

$$= (x_1 + x_2 + \ldots + x_n)\left[\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)^n - x_1 x_2 \cdots x_n\right] \geq 0 ,$$

$$\frac{x_1^{n+1} + x_2^{n+1} + \ldots + x_n^{n+1} - x_1 x_2 \cdots x_n (x_1 + x_2 + \ldots + x_n)}{x_1^{n+1} + x_2^{n+1} + \ldots + x_n^{n+1}} \geq 0 .$$

Hence $\dfrac{x_1^{n+2} - x_1^2}{x_1^{n+2} + x_2^2 + \ldots + x_n^2} + \dfrac{x_2^{n+2} - x_2^2}{x_2^{n+2} + x_3^2 + \ldots + x_1^2} + \ldots +$

$$\frac{x_n^{n+2} - x_n^2}{x_n^{n+2} + x_1^2 + \ldots + x_{n-1}^2} \geq 0 .$$