Making Indirect Reasoning Accessible: A Problem-Based Learning Approach(*)

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Contemporary discussions of goals for mathematics education emphasize the importance of thinking, understanding, reasoning, and problem solving (e.g., National Council of Teachers of Mathematics, 2000). A major goal of mathematics instruction is to help students develop their reasoning skills and gain confidence to reason logically and mathematically. Students should learn that reasoning is fundamental to an understanding of and doing mathematics, and that logical reasoning is the way in which the validity of a mathematical assertion is ensured. One of the ways to reason logically is to construct proofs using indirect methods. Although indirect reasoning is a classic topic, the inherent mathematical structure and reasoning in the *Standards* requires this topic to be given greater importance in the curriculum (NCTM, 1989). Indirect reasoning is fundamental for doing mathematics at all levels. It is also an important idea and approach for conducting research in subject areas such as engineering and science (Maud, 1987). In many cases, use of an indirect proof can provide much more effective mathematical arguments and communication than the use of a direct proof.

However, research shows that a majority of the high school students and preservice teachers do not understand the meaning of indirect proof. For example, in the evaluation study of Precalculus and Discrete Mathematics from the University of Chicago School Mathematics Project, Thompson (1994) found that less than 20% of the students who took Precalculus and Discrete Mathematics from nine participating high schools selected correct answers on

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items involving indirect proof. In another study, Williams (1979) investigated a group of 255 eleventh grade students' understanding of the nature of proof and found that more than 70% of his sample did not understand indirect proof. Many teachers' understanding of an indirect proof is something related to $\sqrt{2}$. When they were asked how an indirect proof is related to $\sqrt{2}$, a majority of them indicated that the use of an indirect proof demonstrated the irrationality of $\sqrt{2}$. However, when they were asked to actually construct an indirect proof of the irrationality of $\sqrt{2}$, many teachers had difficulty in getting started.

Although NCTM (1989, 2000) suggests that only college-bound students need to have experiences in proving mathematical assertions using indirect proof, mathematics teachers should understand the nature and logic of an indirect proof. In order to help school students construct a proof using indirect reasoning, teachers not only need to know the logic and principles underlying an indirect proof but also need to know how to teach students to learn and apply such techniques. We need to seek innovative alternatives to help preservice teachers understand the indirect proof mathematically and pedagogically.

The purpose of this article is to discuss a problem-based learning approach to make an indirect proof meaningful to preservice secondary mathematics teachers. According to the constructivist perspective of learning mathematics, learning is an individual process and each individual responds to learning situations in terms of the meaning he/she has for them (Cobb, 1994). Learners need to experience a constructive process based on his or her own thinking. Using a problem-based learning approach, they actively participate in the processes of knowledge construction and, therefore, make sense of the reasoning processes involved in indirect reasoning in his or her own terms. They become active participants in the creation of knowledge rather than passive receivers of rules and procedures of indirect reasoning. The processes and meaning of indirect reasoning are introduced through problems derived from the learners' familiar world. According to the social perspective of learning mathematics, on the other hand, learners become acculturated by participating in cultural practices. Learning, in this perspective, is a social process in which each individual learns mathematics through social interaction, meaning negotiation, and reaching shared understanding. Problem-based learning approach provides a natural setting for individuals to learn mathematics through social interactions. As they explain and justify their thinking and challenge the explanations of others, they are also engaging in clarification of their own thinking and becoming owners of "knowing."

In this article, the discussion of the problem-based learning approach to make an indirect proof meaningful to preservice secondary mathematics teachers is also based on our experience working with preservice teachers in the past several years and is illustrated using specific activities and examples. The following two assumptions guided the design of activities and examples: (1) We need to use familiar and friendly problems in order to build up preservice teachers' informal ideas of indirect reasoning before the indirect proof is formally introduced; and (2) Once they build up informal ideas, they need to establish the principles and structures of indirect proof.

Building Informal Ideas of Indirect Proof

An individual's current knowledge structure and mental representations play a central role in learning and understanding (Cobb, 1994). Preservice teachers have difficulty with the formal structure of an indirect proof before they become familiar with the ideas in indirect proof. Because of the "indirect" nature, many teachers and students became frustrated with the indirect proof method (Leron, 1985). Thus, we need to build preservice teachers' indirect reasoning ideas using friendly and familiar problem situations. In fact, we all encounter indirect reasoning all the time in our daily lives. The arguments in the following scenarios are examples of using indirect reasoning. They were presented with the following cases and were asked to discuss and explain the reasoning involved.

Case 1. John and Mary are sitting in a library.

John: I've heard from TV that it might be raining today.

Mary: It must be raining right now. If it were not raining, then people coming into the library would be dry, but they are wet.

Case 2. John and Mary drove past a football stadium.

John: Is there any game in this stadium right now?

Mary: I don't think so. If a game were being played right now, the parking lots would be full of cars, but there are no cars in the lots.

In each case, Mary argued that a certain thing is true. In the first case, she argued that "It is raining right now." In order to make this argument, Mary begins by supposing it is not raining. This assumption leads to the conclusion that people coming into the library would be dry based on common sense. This conclusion contradicts the reality that people coming into the library are wet. The contradiction resulted from the fact that Mary assumed it was not raining. Thus, the assumption that it is not raining must be wrong. Similarly, in the second case, Mary argues that "there is no football game in the stadium right now" by supposing that there is a football game in the stadium. Using common sense, if there is a game, the parking lots would be full of cars. This leads to a contradiction since there are no cars in the lots. Therefore, the supposition must be incorrect and there is not a football game in the stadium right now.

After the discussion, these preservice teachers began formulating ideas of indirect reasoning involved in these two examples. Then teachers can move further by proposing the following problem, which is modified from a classical problem used in problem solving research (Rubinstein, 1975).

A woman with her three children met a man at an apartment building.
Woman: "Hi, have not seen you for a long time, how are you?"

Man: "Good to see you. That's right! Have not seen you for a while. Look, your children have grown up so quickly. How old are they now?"

Woman: "The product of their ages is 36."

Man: "Come on, that does not give me enough information to know their ages."

Woman: "The sum of their ages is the same as your apartment number."

Man: "That still does not give me enough information."

Woman: "My oldest child has red hair."

What are the ages of this woman's three children?

Preservice teachers were divided into groups of 3 to work on this problem. Since the product of the children's ages is 36, they quickly realized that there are several possibilities for their ages, including (36, 1, 1); (18, 2, 1); (12, 3, 1), (9, 4, 1); (9, 2, 2); (6, 6, 1), (6, 3, 2), and (4, 3, 3). What puzzled them was the fact that the man's apartment number was unknown. Several groups of teachers even asked for the man's apartment number. However, they did not get an answer to that question. Instead, they were asked a question: "Why do you have to know the man's apartment number?" One of the preservice teachers responded, "If we know the man's apartment number, then we will know their ages since the sum of their ages is the same as the man's apartment number."

The group discussion seemed to inspire their thinking. Possibilities for their ages are (36, 1, 1); (18, 2, 1); (12, 3, 1), (9, 4, 1); (9, 2, 2); (6, 6, 1), (6, 3, 2), and (4, 3, 3). Moreover, according to the woman, the sum of the three ages is the same as the man's apartment number. 36 + 1 + 1 = 38. 18 + 2 + 1 = 21. 12 + 3 + 1 = 16. 9 + 4 + 1 =14. 9 + 2 + 2 = 13. 6 + 6 + 1 =13. 6 + 3 + 2 =11. 4 + 3 + 3 = 10. They realized that the apartment number could be 38, 21, 16, 14, 13, 11, or 10. Discussions continued. After about 25 minutes, one student suddenly spoke loudly: "Indirect reasoning! Indirect reasoning. I think I've got it." Then she explained her thoughts to her group and everybody seemed to be convinced. Other groups also tried to use indirect reasoning to solve the problem and made some progress. There were only five minutes left for the class. They were called together for class discussion. One of the members from the group that figured out the problem first was asked to share the solution.

In their solution, indirect reasoning played an important role. Using indirect reasoning, they eliminated the possibilities of the apartment number as 38, 21, 16, 14, 11, and 10. For example, the teacher reasoned as follows: If the man's apartment number is 11, then the sum of their ages is equal to 11, which can only result from the fact that the ages of the three children are 6, 3, and 2. Therefore, the man would have known the ages of the three children after the woman told him that the sum of their ages is the same as his apartment number. However, the man said he still did not have enough information to know their ages (a contradiction). That means the assumption that the man's apartment number is 11 is incorrect. In other words, the sum of the three ages is not 11. Using the same reasoning, the man's apartment number could not be 14 either. If the apartment number is 14, then the sum of the three ages is equal to 14 (= 9 + 4 + 1). After the woman told him that the sum of their ages is the same as his apartment number, the man would have known the ages of the three children. This contradicts the fact that the man said he still did not have enough information to know their ages. Therefore, the sum of the three ages is not equal to 14. Similarly, the sum of the three ages is not equal to 21 or 38. Naturally, since the sum of the three ages is not equal to 38, 21, 16, 14, 11, or 10, the sum of the three ages must be 13. Thus, there are two possibilities for their ages: (9, 2, 2) or (6, 6, 1). According to the woman, her oldest child has red hair. That means that her oldest children would not be twins. Therefore the ages of her three children are 9, 2, and 2, respectively.

The explanation of the solution was very convincing. In this explanation, use of indirect reasoning was essential to eliminate all but one of the possibilities. To reinforce preservice teachers' understanding of the indirect reasoning, we modified the problem and asked each student to work on the variations as homework. The solution processes are very similar to the original problem. The idea is to use indirect reasoning to eliminate possibilities.

Variation 1: *A woman with her three children met a man at an apartment building. Woman: "Hi, have not seen you for a long time, how are you?"*

> *Man: "Good to see you. That's right! Have not seen you for a while. Look, your children have grown up so quickly. How old are they now?"*
>
> *Woman: "The product of their ages is **72**."*
>
> *Man: "Come on, that does not give me enough information to know their*

ages."

Woman: "The sum of their ages is the same as your apartment number."

Man: "That still does not give me enough information."

Woman: "My oldest child has red hair."

What are the ages of this woman's three children?

Variation 2: *A woman with her three children met a man at an apartment building. Woman: "Hi, have not seen you for a long time, how are you?"*

Man: "Good to see you. That's right! Have not seen you for a while. Look, your children have grown up so quickly. How old are they now?"

Woman: "The product of their ages is 36."

Man: "Come on, that does not give me enough information to know their ages."

*Woman: "**One of their ages is the same as your apartment number**."*

Man: "That still does not give me enough information."

Woman: "My oldest child has red hair."

What are the ages of this woman's three children?

Principles of Indirect Proof

It is not enough for preservice teachers to simply have some ideas of using indirect reasoning. They also need to know the underlying logic. In general, any statement is either true or false. For example, in the first case we introduced earlier, the statement is "It is raining right now." This statement is either true (i.e., consistent with the real fact that it is raining), or false (i.e., it is not raining). It is impossible to have both "raining" and "not raining" at the same time. The principles of an indirect proof are based on the following: (1) Every statement is either true or false (the Law of the Excluded Middle) and (2) No statement can be both true and false (the Law of Noncontradiction). These laws imply that if a statement is true, then the negation of the statement must be false, or vice versa. Therefore, in order to prove a statement is true, it is only necessary to prove that the negation of the statement is false. In order to prove the negation of the statement is false, on needs to show that a contradiction results from assuming the negation of the statement is true.

For example, Mary wanted to justify that the statement "It is raining right now" is true. The negation of the statement is "It is not raining right now." She first assumed that it is not raining. If it is not raining, then people who come into the library would not be wet. This contradicts the fact that people coming into the library are wet. Therefore, the negation of the statement "It is not raining right now" is false. Thus, the statement "It is raining right now" is true.

They were also introduced to some quotations from Sherlock Holmes. Holmes often used indirect reasoning to solve mysteries (Ballew, 1994). The principle he used is the following: With a finite number of possibilities, if we eliminate all but one of the possibilities, then the one which remains, no matter how improbable, must be the truth. This idea was illustrated in the previous Three Children Problem. There are seven different possibilities for the sum of the three children's ages. When 10, 11, 14, 16, 21, and 38 were eliminated, then the sum of the children's ages had to be 13. Students discovered that the principles used in this example are the same as those used in the library scenario. The only difference between the two examples is that in the Three Children Problem there are seven possibilities, and there are only two possibilities in the "rain" problem (either raining or not raining).

Structures of Indirect Proof

After teachers discussed and learned the principles underlying indirect reasoning we turned to discuss the structure of an indirect proof. We used the irrationality of $\sqrt{2}$ to illustrate and discuss the structure in an indirect proof. This example was particularly appropriate because: (1) Teachers were more or less familiar with the connection between indirect proof and $\sqrt{2}$; and (2) We were able to use this opportunity to overcome preservice teachers' misconceptions of indirect proof. Teachers were also separated into groups of 3 and asked to write a detailed convincing proof that "$\sqrt{2}$ is not a rational number." Each group was also asked to identify, analyze, and describe structures of the indirect proof.

Generally, each group was now able to justify that √2 is not a rational number using indirect proof. Therefore, class discussion was focused on important phases in the use of an indirect proof. The discussion was especially effective with the specific examples. Through discussion, the preservice teachers were able to identify the following five important phases of an indirect proof: Orientation, Assumption, Conclusion, Contradiction, and Verification.

<u>Orientation.</u> In an indirect proof, an individual first has to understand the problem. Proof is a specific kind of problem solving. It requires comprehension of what is given and what needs to be proved. In particular, an individual has to identify the statement that needs to be proved. In some cases, what has to be proved is very straightforward. In the example of the irrationality of √2, what needs to be proved is very clear: Prove √2 is not a rational number. However, some problems are not always so straightforward, such as in the example of the Three Children Problem.

<u>Assumption.</u> After the statement which is to be proved is identified, an assumption needs to be made. The assumption is the negation of the statement that has to be proved. For example, in the case of the irrationality of √2, the assumption is that "√2 is a rational number."

<u>Conclusion.</u> Once the assumption is made, then one starts to make a series of inferences and reasoning which lead to a conclusion. For example, if √2 is a rational number, then √2 can be expressed as a/b where a and b are positive whole numbers, $b \neq 0$, and a and b are reduced without a common factor other than 1. That is, $\sqrt{2} = a/b$. If we square both sides, we have $2b^2 = a^2$. This implies that a^2 has a factor of 2. Therefore, a has a factor of 2. We can then write $a = 2n$ (n is a positive whole number). Thus, we have $2b^2 = 4n^2$, i.e., $b^2 = 2n^2$. In the same way, it follows that b has a factor of 2. Thus the conclusion is: a and b both have a factor of 2.

<u>Contradiction.</u> The fourth phase in an indirect proof is to find a

contradiction. Usually, the contradiction is found between a known fact or information and the conclusion resulting from the assumption. In the example of irrationality of √2, the conclusion is that *a* and *b* both have a common factor of 2, which contradicts the assumed fact that *a* and *b* are reduced without a common factor other than 1.

Verification. The last phase in an indirect proof is to state the conclusion after a contradiction is found. Because of the contradiction, the assumption that √2 is a rational number must be false. Thus, we have provided a valid argument that √2 is not a rational number.

A through discussion of the phases of indirect proof was helpful for preservice mathematics teachers in their understanding of the principles of an indirect proof and forming the mental models of indirect proof. Many teachers indicated that they became more comfortable about introducing the indirect proof to their students. We also asked preservice teachers to construct an indirect proof of the irrationality of √3 based on the five phases discussed above.

Direct and Indirect Proofs

After they discussed and understood the principles and structures of indirect proof, a natural question many preservice teachers asked was: "What do I tell my students when to use direct proof and when to use indirect proof?" There is no definite answer to this question. Like working forwards and working backwards, there are no definite rules on when to use a particular strategy. However, a general rule is that when one has been unable to find a direct proof or a direct proof is very time-consuming, we frequently turn to an indirect proof. For example, it would be almost impossible to use a direct proof to prove that √2 is an irrational number.

In other cases, one can use either a direct or an indirect proof, but an indirect proof may be more communicable. In this direction, teachers found the following problem enlightening: "Let a_1, a_2, a_3, a_4, …, $a_{(2n+1)}$ be an arrangement of the numbers 1, 2, 3, 4, …, $(2n + 1)$. Show that the product of

$(1 - a_1)(2 - a_2)(3 - a_3)(4 - a_4) \ldots [(2n + 1) - a_{(2n+1)}]$ is even." We can use both direct and indirect proofs to show that the product of $(1 - a_1)(2 - a_2)(3 - a_3)(4 - a_4) \ldots [(2n + 1) - a_{(2n+1)}]$ is even. Preservice teachers were asked to provide both direct and indirect proofs for this problem.

Direct proof. For the product to be even, at least one factor must be even. Every number is either even or odd. Totally, there are four possibilities for each difference: (1) odd – odd = even, (2) odd – even = odd, (3) even – odd = odd, and (4) even – even = even. Only for odd – even or even – odd does one get odd differences. Since we have a total of $(2n+1)$ numbers, with n even numbers and $(n+1)$ odd numbers, we cannot have all even – odd or odd – even. Therefore, one of the differences must be either odd – odd or even – even. This means that one of the differences must be even. Thus the product of $(1 - a_1)(2 - a_2)(3 - a_3)(4 - a_4) \ldots [(2n + 1) - a_{(2n+1)}]$ is even.

Indirect proof. The statement we need to prove is that the product of $(1 - a_1)(2 - a_2)(3 - a_3)(4 - a_4) \ldots [(2n + 1) - a_{(2n+1)}]$ is even (orientation). Let's assume the product of $(1 - a_1)(2 - a_2)(3 - a_3)(4 - a_4) \ldots [(2n + 1) - a_{(2n+1)}]$ is not even (assumption). Then each difference term must be odd.

That is

$$
\begin{aligned}
(1 - a_1) &= \text{odd number} \\
(2 - a_2) &= \text{odd number} \\
(3 - a_3) &= \text{odd number} \\
(4 - a_4) &= \text{odd number} \\
&\cdots\cdots \\
(2n + 1) - a_{(2n+1)} &= \text{odd number}
\end{aligned}
$$

If we add terms on the left side of the equal sign together and add terms on the right side of the equal sign, then they should still be equal. That is $(1 - a_1) + (2 - a_2) + (3 - a_3) + (4 - a_4) + \ldots + [(2n + 1) - a_{(2n+1)}] =$ odd + odd + odd + … + odd. That is, 0 = the sum of $(2n + 1)$ odd numbers. Since the sum of the $(2n + 1)$ odd numbers is equal to an odd number, we have 0 = an odd number (conclusion). This conclusion itself contains a contradiction because 0 is not equal to an odd number (contradiction). Therefore, our initial assumption that the product of $(1 - a_1)(2 - a_2)(3 - a_3)(4 - a_4) \ldots [(2n + 1) - a_{(2n+1)}]$ is not an even number is incorrect. Thus, we proved that the product of $(1 - a_1)(2 - a_2)(3 - a_3)(4 - a_4) \ldots [(2n + 1) - a_{(2n+1)}]$ is even (verification).

After they had done this problem, many preservice teachers felt that indirect proof appeared to be more "direct" than direct proof. Using direct proof, they seemed to have difficulty making explicit arguments. Interestingly, they felt that the following argument was not explicit and convincing: *Since we have a total of (2*n*+1) numbers, with* n *even numbers and (*n*+1) odd numbers, we cannot have all even – odd or odd – even. Therefore, one of the differences must be either odd – odd or even – even.* However, by using an indirect proof, they felt they had a good handle on each phase of the proof. Each step seemed to be very explicit and convincing.

This example showed them the value of indirect proof. In particular, this example suggests that proving $\sqrt{2}$ is an irrational number is not the only place to use indirect reasoning. Since we learn mathematics if we are actively engaged in creating not only solution strategies but problems that require certain techniques (Moses, Bjork, & Goldenberg, 1990), we asked preservice teachers to design mathematical and non-mathematical problems which make use of an indirect proof. Here are a few examples:

Example 1: The numbers 1 through 9 can be placed into a 3 by 3 square board so that the sum of the numbers in each row, column, and diagonal is 15. Prove that 9 cannot be placed on the up-right corner.

Example 2: n is an integer. If n^2 is an odd number, n must be an odd number.

Example 3: In a triangle, there is one and only one altitude to each side.

In addition, they were also provided opportunities to evaluate various "proofs" to a mathematical statement. They were asked to judge if each of the proposed proofs is correct. If a proof is incorrect, they are asked to provide a correct proof. If it is correct, they are asked to provide the logic behind for the given proof. This kinds of activities facilitated their understanding of what constitutes a proof and how one constructs a direct or indirect proof. Below is an example.

Example: Consider the following statement: Let m be an integer. If m is even, then m^2 is even. A group of college freshmen proposed the following proofs. Evaluate if each of these proposed proofs is correct. If a proof is incorrect, please provide a correct proof. If it is correct, please provide the logic behind for the given proof.

Proposed proof 1: Assume m is not even. Then m is odd. Thus, there exists an integer k such that $m = 2k + 1$. Therefore, $m^2 = (2k + 1)^2 = 4k^2 + 4k + 1$, which is odd. Thus if m is odd, then m^2 is odd. Therefore, If m is even, then m^2 is even.

Proposed proof 2: Since m is even, when $m = 2$, $2^2 = 4$, which is even. When $m = 4$, $4^2 = 16$, which is even. When $m = 6$, $6^2 = 36$, which is even. Therefore, if m is even, then m^2 is even.

Proposed proof 3: Since m is even, it can be expressed as $m = 2k$ (for some integer k). $m^2 = (2k)^2 = 4k^2$. Since $4k^2$ is even, m^2 is always even when m is even.

Proposed proof 4: If m^2 is not even, then m^2 is odd. Therefore m^2 can be expressed as $m^2 = 2b + 1$ for some integer b. $m^2 - 1 = 2b$. $(m - 1)(m + 1) = 2b$. Since m is even, it follows that $(m - 1)$ is odd and $(m + 1)$ is odd. Thus, $(m - 1)(m + 1)$ is odd, which contradicts the fact that $(m - 1)(m + 1) = 2b$, which is even. Therefore, m^2 must be even.

Proposed proof 5: If n is even, n can be written as $n = 2m$ for some integer m. $n^2 = (2m)^2 = 4m^2$. Therefore, n^2 is even. This contradicts with the fact that n^2 is odd. Thus, n^2 must be odd.

Conclusion

It is necessary mathematics teachers to understand the meaning of indirect reasoning. A problem-based learning approach seems to be effective for making indirect reasoning accessible to preservice teachers. Preservice teachers are able to learn and understand indirect reasoning by examining problem situations from everyday life and mathematics itself in which such reasoning is consciously or unconsciously used. Preservice mathematics teachers have experienced the success to transfer such activities to a higher plane of mathematical thoughts in which conjectures are verified or counterexamples are constructed.

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