Solving Triangles by the Cosine Rule and Constructing Integer-Sided 60° or 120° Triangles

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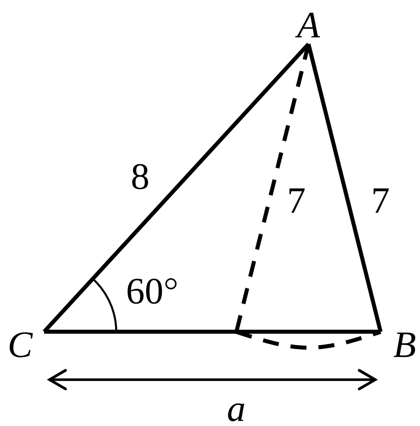
Solving Triangles by the Cosine Rule

To solve a triangle with two sides and one non-included angle given (SSA), the standard treatment in textbooks is using the sine rule. However, using the cosine rule in this case has advantages over the sine rule. If there are two solutions for the triangle, many students using the sine rule may easily forget the possibility of the second solution. If the cosine rule is used, both solutions would be generated simultaneously by solving a quadratic equation, as shown in the following example.

Example 1

In ΔABC, $c = 7$, $b = 8$ and $\angle C = 60°$, solve ΔABC.



Solution

Using the cosine rule,

$$\begin{aligned} a^2 + 8^2 - 2 \cdot a \cdot 8 \cdot \cos 60° &= 7^2 \\ a^2 - 8a + 15 &= 0 \\ (a-3)(a-5) &= 0 \\ a &= 3 \quad \text{or} \quad 5 \end{aligned}$$

When $a = 3$, by the cosine rule,

$$\angle C = \cos^{-1}\frac{7^2+3^2-8^2}{2\cdot 7\cdot 3} = 98.21^\circ \quad \text{(to 2 d.p.)}$$

$$\angle B = 180^\circ - 60^\circ - 98.21^\circ = 21.79^\circ \quad \text{(to 2 d.p.)}$$

When $a = 5$, by the cosine rule,

$$\angle C = \cos^{-1}\frac{7^2+5^2-8^2}{2\cdot 7\cdot 5} = 81.79^\circ \quad \text{(to 2 d.p.)}$$

$$\angle B = 180^\circ - 60^\circ - 81.79^\circ = 38.21^\circ \quad \text{(to 2 d.p.)}$$

The solutions are $a = 3$, $\angle C = 98.21^\circ$, $\angle B = 21.79^\circ$ or $a = 5$, $\angle C = 81.79^\circ$, $\angle B = 38.21^\circ$.

Another advantage of using the cosine rule is that the cosine function tells us whether the angle involved is acute or obtuse while sine function does not. In the above example, in the case $a = 3$, if the sine rule is used to find $\angle C$ we would have $\sin\angle C = \dfrac{8\sin 60^\circ}{7}$ and we are unable to tell whether $\angle C = 81.79^\circ$ or $180^\circ - 81.79^\circ = 98.21^\circ$, and most students would just give the wrong answer $\angle C = 81.79^\circ$ without second thoughts. Instead, the cosine rule gives $\cos\angle C = \dfrac{7^2+3^2-8^2}{2\cdot 7\cdot 3} = -\dfrac{1}{7}$ which suggests that $\angle C$ must be obtuse.

Therefore teachers could remind students the possibility of using the cosine rule to solve SSA triangles or to find some unknown angles, which is more straightforward and safe.

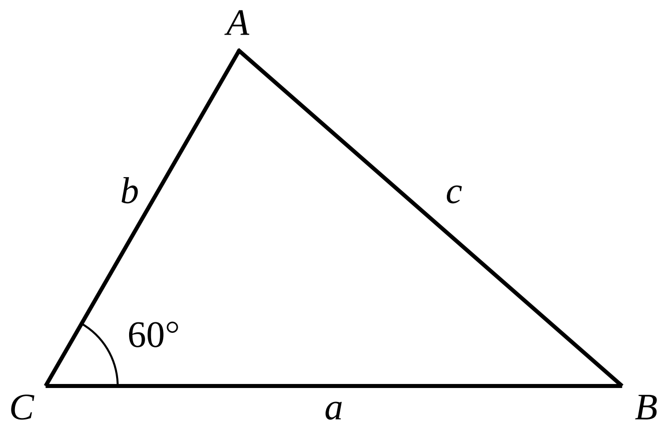
Constructing Integer-Sided 60° or 120° Triangles

To give students more practice about solving SSA triangles with two solutions, I have tried to construct some 60° triangles of integer sides.

Consider ΔABC with b, c given $\angle C = 60^\circ$. By the cosine rule,

$$a^2 - 2ab\cos 60^\circ + b^2 = c^2$$

$$a^2 - ab + b^2 = c^2 \quad \text{.................................. (*)}$$



So I have to find positive integers a , b and c satisfying (*). The solutions are given by:

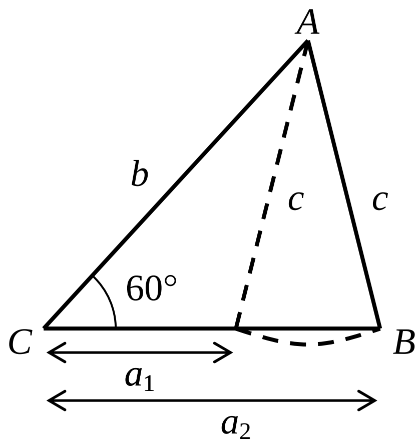
$a = m^2 - n^2$, $b = m^2 + 2mn$, $c = m^2 + mn + n^2$

where m , n are positive integers with $m > n$. ([1])

The following are some solutions obtained by putting some values of m and n :

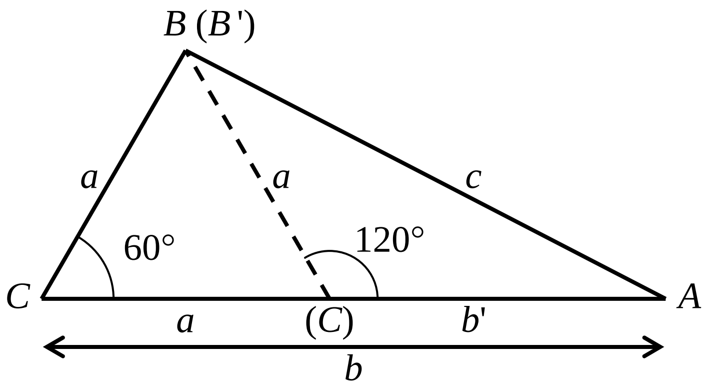
m	n	$a = m^2 - n^2$	$b = m^2 + 2mn$	$c = m^2 + mn + n^2$ $= \sqrt{a^2 - ab + b^2}$
2	1	3	8	7
3	2	5	21	19
3	1	8	15	13
4	3	7	40	37
5	4	9	65	61
6	5	11	96	91
7	6	13	133	127

Note that (*) is a quadratic equation in a . Since the sum of roots = b , if $a = a_1$ is a solution of (*), so is $a = b - a_1$. Hence the above table gives the following SSA triangles (b , c are given and $\angle C = 60°$) with 2 solutions $a = a_1$ or a_2 .



a_1	a_2 (= $b - a_1$)	b	c
3	5	8	7
5	16	21	19
7	8	15	13
7	33	40	37
9	56	65	61
11	85	96	91
13	120	133	127

Note also that once an integer-sided 60° triangle is constructed, a corresponding integer-sided 120° triangle can be obtained immediately. In the figure, if ΔABC is a triangle of integer sides a, b, c ($b > a$) and $\angle C = 60°$, then $\Delta AB(C)$ would be a triangle of sides a, c, $b' = b - a$ and $\angle C = 120°$. Since $a = m^2 - n^2$, $b = m^2 + 2mn$, $c = m^2 + mn + n^2$, we have $a = m^2 - n^2$, $b' = 2mn + n^2$, $c = m^2 + mn + n^2$, which form a triangle with $\angle C = 120°$ and satisfy $c^2 = a^2 + ab' + b'^2$. The following table gives the sides of some of these triangles.



a	b	b'	c
3	8	5	7
5	21	16	19
8	15	7	13
7	40	33	37
9	65	56	61
11	96	85	91
13	133	120	127

It is interesting to know that the integer solutions of (*) are also useful in constructing cubic polynomials whose zeros and the zeros of whose first derivatives are integers. ([2]) Consider $f(x) = x(x - 3a)(x - 3b)$, where a and b are integers. The roots of $f'(x) = 0$ are given by $x = a + b \pm \sqrt{a^2 - ab + b^2}$. Therefore if $a^2 - ab + b^2 = c^2$ is a complete square, $f(x)$ would have integer zeros $0, 3a, 3b$ and its first derivative has integer zeros $a + b \pm c$. From the above, some possible values of (a, b) are $(3, 8)$, $(5, 21)$, $(8, 15)$ which give $f(x) = x(x - 9)(x - 24)$, $f(x) = x(x - 15)(x - 63)$, $f(x) = x(x - 24)(x - 45)$ as examples of these polynomials.

References

1. K.R.S. Sastry, *Natural Number Solutions to* $3(p^4 + q^4 + r^4 + s^4) = (p^2 + q^2 + r^2 + s^2)^2$, Mathematics and Computer Education V34 (2000) pp.6 – 11.
2. C.H. Fima, *Constructing Integer Polynomials and Triangles*. Teaching Mathematics and Its Applications Vol.9 (1990) pp.171 – 174.