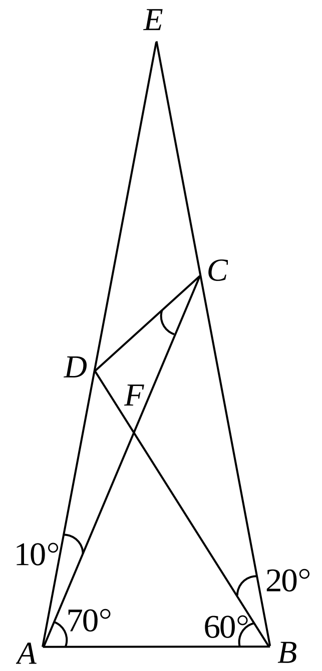
"World's Hardest Easy Geometry Problem"

In the figure, find $\angle ACD$.



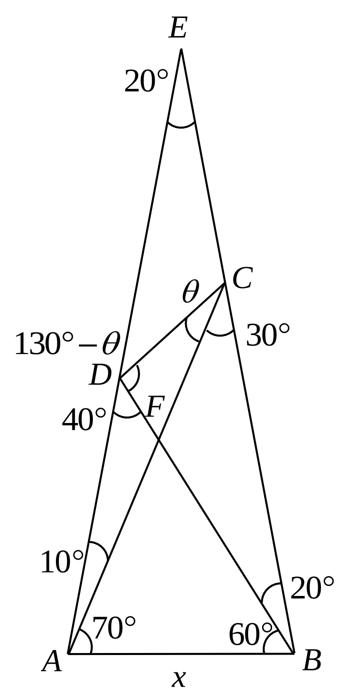
I. Construction by compass and rulers

II. Deductive approach:

Solutions:

Let $\angle ACD = \theta$ and $AB = x$.

By ($\angle$ sum of Δ), we have the following angles as shown in the figure:



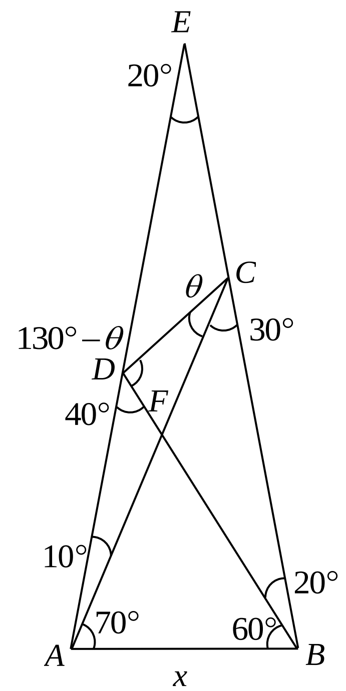
Method 1 (Purely Trigonometry)

In ΔABD, $\dfrac{BD}{\sin 80°} = \dfrac{x}{\sin 40°}$

$$BD = \frac{x \sin 80°}{\sin 40°} = \frac{x(2\sin 40° \cos 40°)}{\sin 40°} = 2x\cos 40° = 2x \sin 50°$$

In ΔABC, $\dfrac{BC}{\sin 70°} = \dfrac{x}{\sin 30°}$

$$BC = \frac{x \sin 70°}{\left(\frac{1}{2}\right)} = 2x \sin 70°$$



In ΔBCD, $\dfrac{BC}{\sin(130°-\theta)}=\dfrac{BD}{\sin(30°+\theta)}$

$$\frac{BC}{BD}=\frac{\sin(130°-\theta)}{\sin(30°+\theta)}$$

$$\frac{2x\sin 70°}{2x\sin 50°}=\frac{\sin[180°-(130°-\theta)]}{\sin(30°+\theta)}$$

$$\frac{\sin 70°}{\sin 50°}=\frac{\sin(50°+\theta)}{\sin(30°+\theta)}$$

$$\sin 70°\sin(30°+\theta)=\sin 50°\sin(50°+\theta)$$

$$-\frac{1}{2}[\cos(100°+\theta)-\cos(40°-\theta)]=-\frac{1}{2}[\cos(100°+\theta)-\cos\theta]$$

$$\cos(40°-\theta)=\cos\theta$$

$\theta=360n°\pm(40°-\theta)$, where n is an integer.

$2\theta=360n°+40°$ or $360n°-40°=0°$ (rej.)

$$\theta=180n°+20°$$

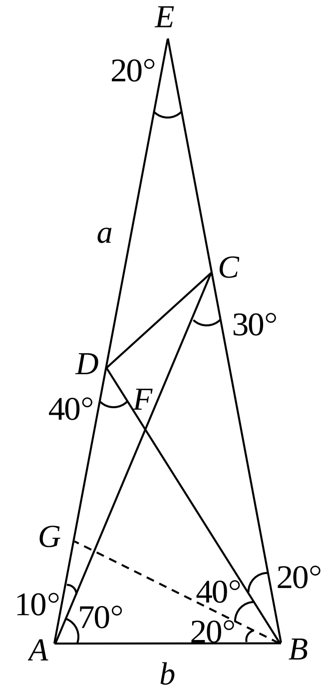
$0°<\theta<130°$, $\therefore \theta=20°$.

Remarks:

1. It is the shortest proof but not an elegant one. Not much insight is developed in this method. One may not know under what circumstances we should interchange $\sin\theta$ and $\sin(180°-\theta)$.

2. Yet, trigonometry establishes the relationship between angles and lengths of sides in geometry. This helps avoid construction of extra straight lines.

<u>Method 2 (Pseudo Geometric Approach)</u>



Let $DE = a$ and $AB = b$

$\angle DEB = \angle DBE = 20°$,

$\therefore DE = DB = a$ (sides opp. eq. $\angle$ s)

Let G be a point on DA so that $\angle GBD = 40°$.

$\therefore \angle GDB = \angle GBD = 40°$

$\therefore GD = GB$ (sides opp. eq. $\angle$ s)

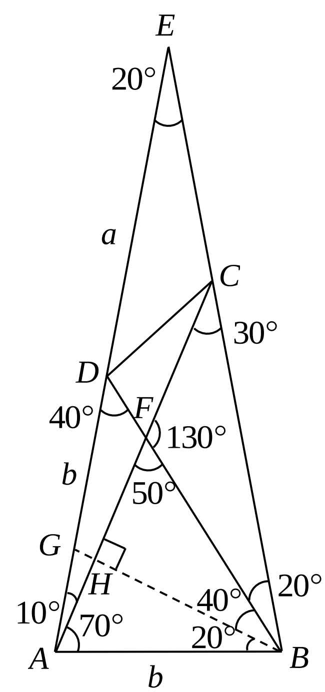
$\angle GBA = 60° - 40° = 20°$

$\angle AGB = 180° - 20° - (10° + 70°) = 80°$ ($\angle$ sum of Δ)

$\angle GAB = \angle AGB = 80°$

$\therefore GB = AB = b$ (sides opp. eq. $\angle$ s)

$\therefore GD = GB = AB = b$



Let H be the point of intersection of AC and BG.

$\angle CHB = 180° - 30° - (20° + 40°) = 90°$ ($\angle$ sum of Δ)

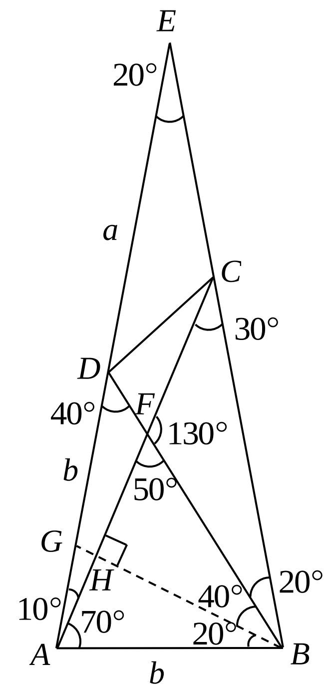
$\angle HFB = 180° - 90° - 40° = 50°$ ($\angle$ sum of Δ)

$\angle CFB = 180° - 50° = 130°$ (adj. $\angle$ s on st. line)

$$\frac{HB}{AB} = \sin 70°$$

$$HB = b \sin 70°$$

$$\frac{HB}{CB} = \sin 30° = \frac{1}{2}$$



$$CB = 2b\sin 70°$$

$$\frac{HB}{FB} = \sin 50°$$

$$FB = \frac{b\sin 70°}{\sin 50°}$$

ΔGBD is isos.

$$\therefore \frac{\left(\frac{BD}{2}\right)}{GD} = \cos 40° = \sin 50°$$

$$a = 2b\sin 50°$$

ΔABE is isos.,

$$\therefore \frac{\left(\frac{AB}{2}\right)}{BE} = \sin\left(\frac{20°}{2}\right) = \sin 10°$$

$$BE = \frac{b}{2\sin 10°}$$

$$CE = BE - BC = \frac{b}{2\sin 10°} - 2b\sin 70°$$

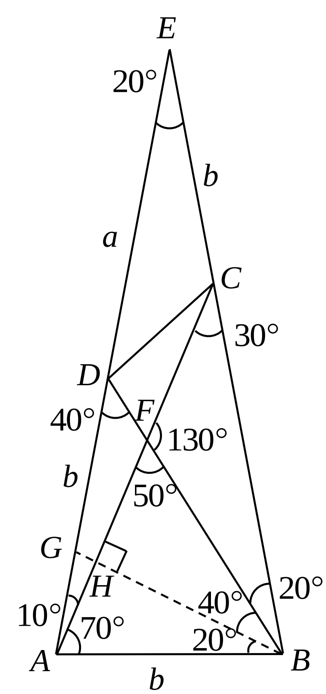
$$CE = b\left(\frac{1}{2\sin 10°} - 2\sin 70°\right) = \frac{b(1-4\sin 70°\sin 10°)}{2\sin 10°} = \frac{b[1-2(\cos 60° - \cos 80°)]}{2\sin 10°}$$

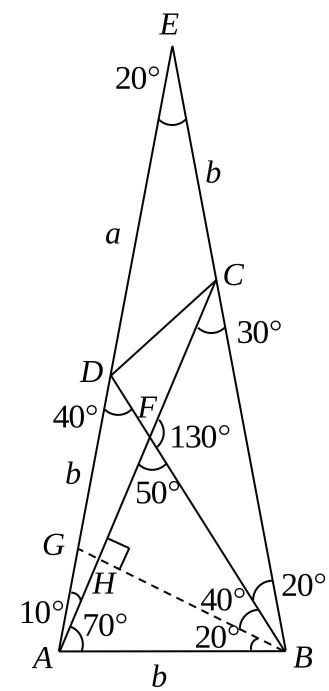
$$CE = \frac{2b\cos 80°}{2\sin 10°} = \frac{b\sin 10°}{\sin 10°} = b$$

In ΔCED & ΔFBC,

$\angle CED = \angle FBC = 20°$ (proved)

$$\frac{ED}{CE} = \frac{2b\sin 50°}{b} = 2\sin 50°$$





$$\frac{BC}{FB} = \frac{2b\sin 70°}{\left(\frac{b\sin 70°}{\sin 50°}\right)} = 2\sin 50°$$

$$\therefore \frac{ED}{CE} = 2\sin 50° = \frac{BC}{FB}$$

$\therefore \Delta CED \sim \Delta FBC$ (ratio of 2 sides, inc. $\angle$)

$\therefore \angle ECD = \angle BFC = 130°$ (corr. $\angle$s, $\sim \Delta$ s)

$\angle ACD = 180° - 30° - 130° = 20°$ (adj. $\angle$s on st. line)

<u>Remarks:</u>

1. It is not natural to guess that $\Delta CED \sim \Delta FBC$ with limited given conditions.

2. By constructing isosceles ΔGBD, we fortunately get the right angle $\angle CHB$. Hence we get all the angles $10°, 20°, 30°, \quad , 80°, 90°$ and $\Delta GBA \sim \Delta AEB$, though it is not used in the proof.

3. This method yields some more trigonometric identities:

(a) $CE = \dfrac{b}{2\sin 10°} - 2b\sin 70° = b$

$$\therefore \frac{1}{2\sin 10°} - 2\sin 70° = 1$$

(b) $\dfrac{\left(\frac{GA}{2}\right)}{b} = \sin\left(\dfrac{20°}{2}\right) = \sin 10°$

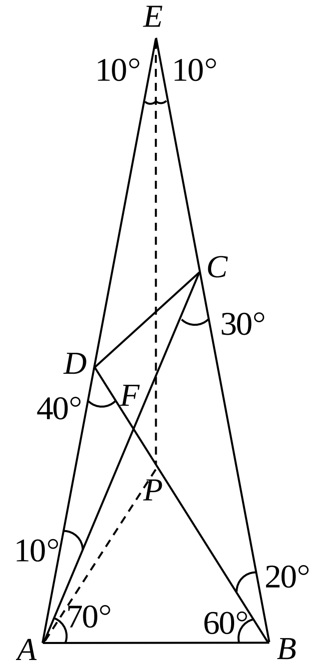
$\therefore GA = 2b\sin 10°$

from $EA = EB$, we have $2b\sin 50° + b + 2b\sin 10° = b + 2b\sin 70°$

i.e. $\sin 10° + \sin 50° = \sin 70°$ or equivalently, $\cos 40° + \cos 80° = \cos 20°$.

4. Indeed students may use calculator to obtain the identity in point 3, hence students with knowledge of geometry (up to congruent and similar triangles) and trigonometric RATIOs only are able to understand (but not necessarily to give!) the proof.

Method 3 (Purely Geometric Approach)



Let P be a point on DB such that EP is the angle bisector of $\angle AEB$.

Construct AP.

In ΔAEP & ΔBEP,

$EP = EP$ (common)

$\angle AEP = \angle BEP = 10°$ (by construction)

$\angle EAB = \angle EBA = 80°$

$\therefore AE = BE$ (sides opp. eq. $\angle$ s)

$\therefore \Delta AEP \cong \Delta BEP$ (SAS)

$\therefore \angle EAP = \angle EBP = 20°$ (corr. $\angle$ s, $\cong \Delta$ s)

$\angle CAP = 20° - 10° = 10°$

$\angle PAB = 70° - 10° = 60°$

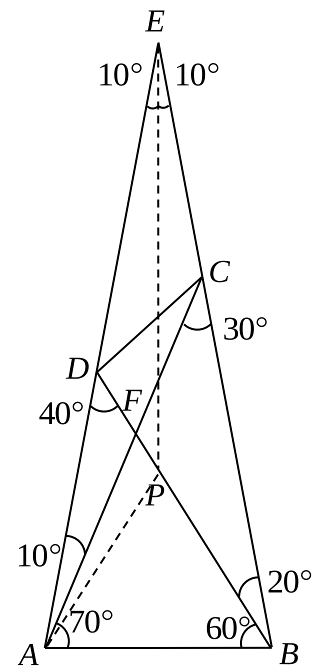
$\angle APB = 180° - 60° - 60° = 60°$ ($\angle$ sum of Δ)

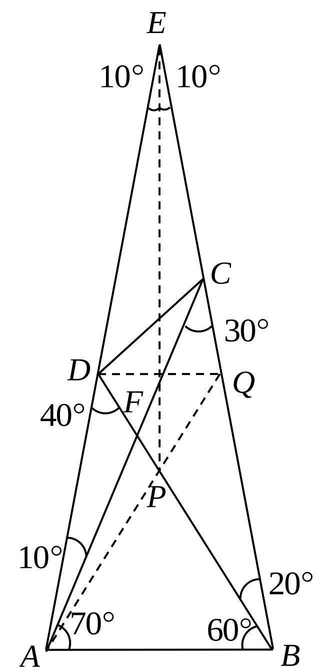
$\therefore \Delta APB$ is an equil. Δ

$\therefore AP = PB = AB$ (def. of equil. Δ)

Let them be y.

In ΔAEC & ΔEBP,





$\angle EAC = \angle BEP = 10°$ (proved)

$\angle AEC = \angle EBP = 20°$ (proved)

$AE = EB$ (proved)

$\therefore \Delta AEC \cong \Delta EBP$ (ASA)

$\therefore EC = BP = y$ (corr. sides, $\cong \Delta$ s)

Extend *AP* to meet *BE* at *Q*. Construct *DQ*.

$\angle DPQ = \angle APB = 60°$ (vert. opp. $\angle$ s)

In ΔDAB & ΔQBA,

$\angle DAB = \angle QBA = 80°$ (given)

$AB = BA$ (common)

$\angle DBA = \angle QAB = 60°$ (proved)

$\therefore \Delta DAB \cong \Delta QBA$ (ASA)

$\therefore DB = QA$ (corr. sides, $\cong \Delta$ s)

Let them be *x*.

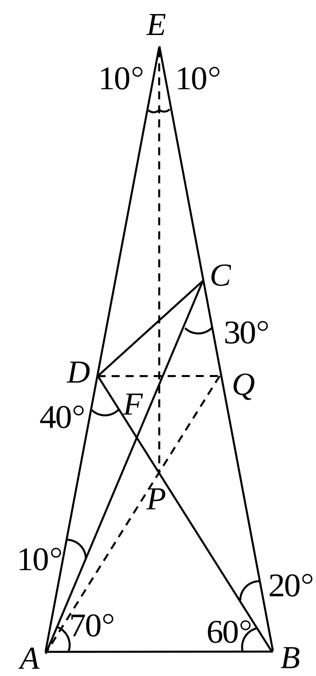
$DP = DB - PB = x - y = QA - PA = QP$

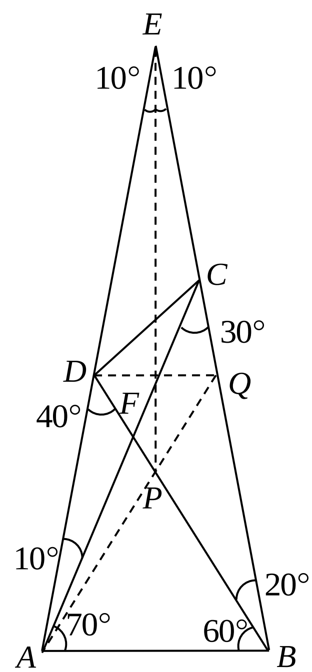
$\therefore \angle PDQ = \angle PQD$ (base $\angle$ s, isos. Δ)

$= \dfrac{180° - 60°}{2} = 60°$ ($\angle$ sum of Δ)

$\therefore \Delta DPQ$ is an equil. Δ

$\therefore DQ = QP = DP = x - y$ (def. of equil. Δ)





$\angle PQD = \angle QAB = 60°$

$\therefore DQ // AB$ (alt. $\angle$s eq.)

$\therefore \angle EQD = \angle EBA = 80°$ (corr. $\angle$s, $DQ // AB$)

$\angle EDQ = 180° - 20° - 80° = 80°$ ($\angle$ sum of Δ)

$\angle DEB = \angle DBE = 20°$

$\therefore ED = DB = x$ (sides opp. eq. $\angle$s)

$\angle EDQ = \angle EQD = 80°$

$\therefore EQ = ED = x$ (sides opp. eq. $\angle$s)

$\therefore CQ = EQ - EC = x - y = DQ$

$\therefore \angle QCD = \angle QDC$ (base $\angle$s, isos. Δ)

$= \dfrac{180° - 80°}{2} = 50°$ ($\angle$ sum of Δ)

$\therefore \angle ACD = 50° - 30° = 20°$

<u>Remarks:</u>

1. This is the most famous proof. All the other proofs that can be found on the web employ the same construction of straight lines.

2. It is a natural way to divide the isosceles triangle along the axis of symmetry. By doing so, we are lucky to obtain equilateral triangles and parallel lines.

3. As it is purely deductive geometric approach, the proof is long and complicated, but an elegant one.

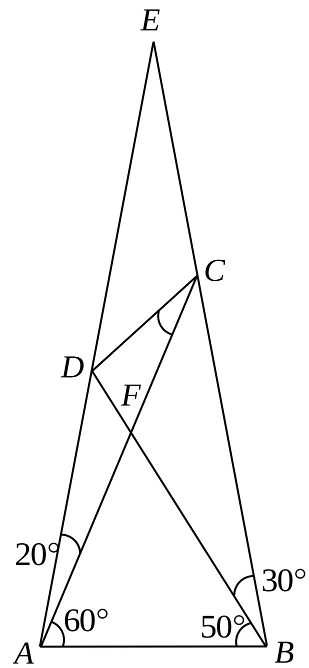
Teaching in Junior Secondary Level

The pedagogy of "Cooperative Learning" is adopted. Practically it is done in the following way: The problem is divided into 5 parts for 5 expert groups (A, B, C, D, E) of students to work on. After that, 1 student from each expert group will form a STAD group, and they will combine the results together. See the appendix of the worksheets.

Further Discussion

1. "World Second Hardest Easy Geometry Problem"

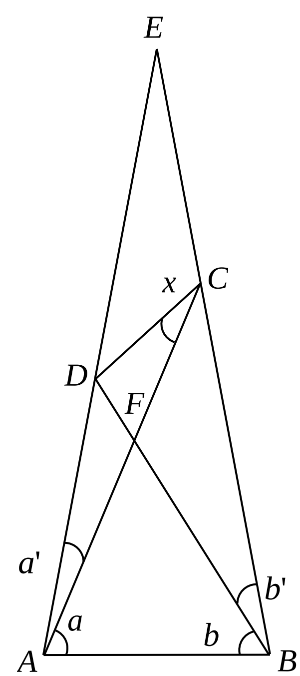
In the figure, find $\angle ACD$.



If we try to generalize the situation as follows:

Consider ΔABE with some angles marked as shown in the figure. A natural question is: "Is there any general relation between the marked angles?"

Similar to Method 1 above, we apply sine law several times.



In ΔABD, $\dfrac{AD}{\sin b} = \dfrac{AB}{\sin(180° - a - a' - b)}$

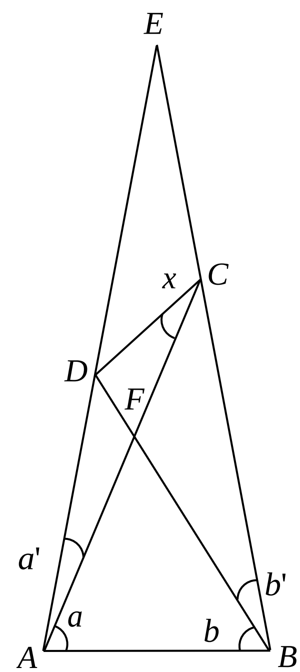
$$\frac{AD}{AB} = \frac{\sin b}{\sin(a + a' + b)} \quad \ldots\ldots(1)$$

In ΔABC, $\dfrac{AC}{\sin(b + b')} = \dfrac{AB}{\sin(180° - a - b - b')}$

$$\frac{AB}{AC} = \frac{\sin(a + b + b')}{\sin(b + b')} \quad \ldots\ldots(2)$$

In ΔADC, $\dfrac{AD}{\sin x} = \dfrac{AC}{\sin(180° - a' - x)}$

$$\frac{AC}{AD} = \frac{\sin(x + a')}{\sin x} \quad \ldots\ldots(3)$$



$(1)\times(2)\times(3)$ $\dfrac{AD}{AB}\times\dfrac{AB}{AC}\times\dfrac{AC}{AD} = \dfrac{\sin b}{\sin(a + a' + b)}\times\dfrac{\sin(a + b + b')}{\sin(b + b')}\times\dfrac{\sin(x + a')}{\sin x}$

$$\frac{\sin(x + a')}{\sin x} = \frac{\sin(a + a' + b)\times\sin(b + b')}{\sin(a + b + b')\times\sin b} \quad \ldots\ldots(*)$$

The following table shows some solutions of equation (*).

a	a'	b	b'	x
75	25	40	30	30
66	22	46	30	30
60	20	50	30	30
39	13	64	30	30
30	10	70	30	30
21	7	76	30	30
6	2	86	30	30

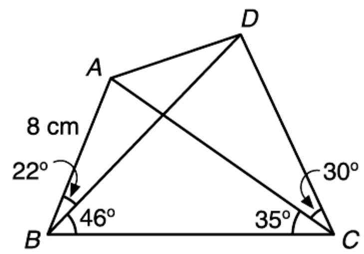
Some patterns of the solution are observed. If we start with $(a_0, a'_0, b_0) = (75, 25, 40)$, while keeping $b' = 30$ and $x = 30$ unchanged, then we can generate the the solutions of (*) by replacing (a_0, a'_0, b_0) by $(a_0 - 3k, a'_0 - k, b_0 + 2k)$, where $-5 < k < 25$.

2. In Class Example (DSE Application of Trigonometry in 2–dimensional Problems):

The figure shows a quadrilateral *ABCD*, where $AB = 8$ cm, $\angle ABD = 22°$, $\angle DBC = 46°$, $\angle ACB = 35°$ and $\angle ACD = 30°$.

(a) Find the length of *BC*.

(b) Find the length of *AD*.

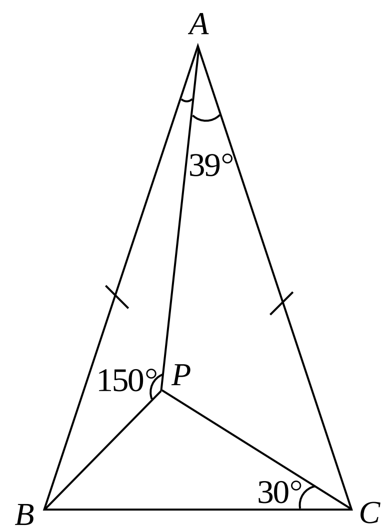


3. IMO Preliminary Selection Contest Hong Kong 2014

Question 9

ΔABC is isosceles with $AB = AC$. P is a point inside ΔABC so that $\angle BCP = 30°$, $\angle APB = 150°$ and $\angle CAP = 39°$. Find $\angle BAP$.

(1 mark)



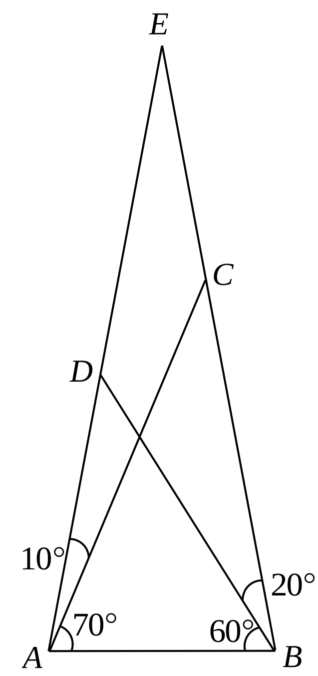
Answers:

1.) 30°   2.) (a) 13.6 cm   (b) 6.51 cm   3.) 13°

Appendix:

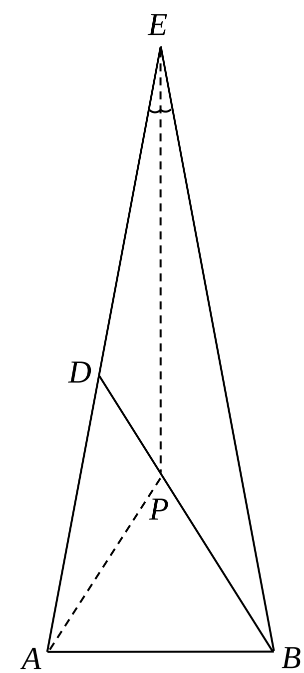
F.2 Math CL Geometry (Expert Summary Group 1)

(a) Prove that $EA = EB$.

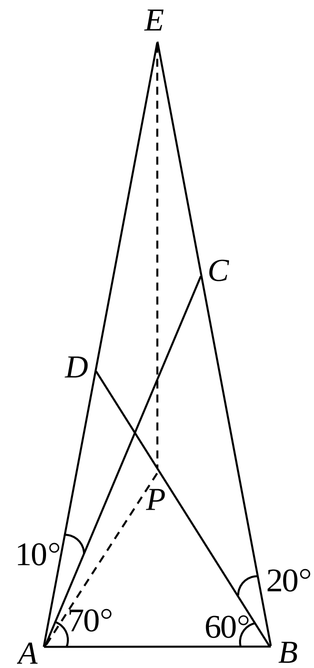


Let *P* be a point on *DB* so that *EP* is the angle bisector of $\angle AEB$. Construct *AP*.

(b) Using (a), prove that $\Delta AEP \cong \Delta BEP$.



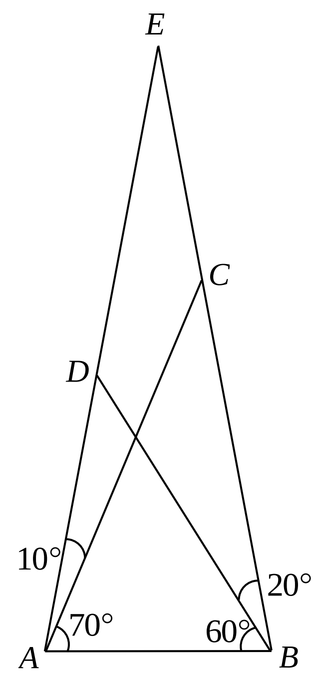
(c) Using (b), find $\angle CAP$ and $\angle PAB$.



(d) Using (c), prove that ΔAPB is an equilateral triangle.

F.2 Math CL Geometry (Expert Summary Group 2)

(a) Find $\angle AEC$.

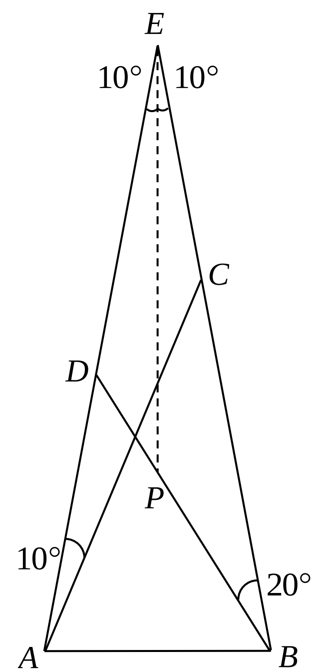


(b) Using (a), prove that $DE = DB$.

Let P be a point on DB so that EP is the angle bisector of $\angle AEB$.

It is given that $AE = EB$. (**proved by Expert Group 1**)

(c) Using (a), prove that $\Delta AEC \cong \Delta EBP$.



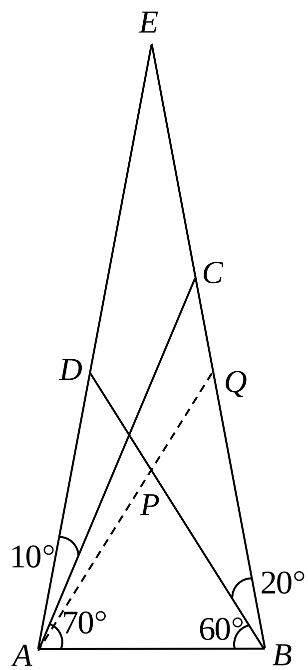
(d) Name the corresponding side of BP.

F.2 Math CL Geometry (Expert Summary Group 3)

Let P be a point on DB so that ΔAPB is an equilateral triangle. (**from Expert Group 1**)

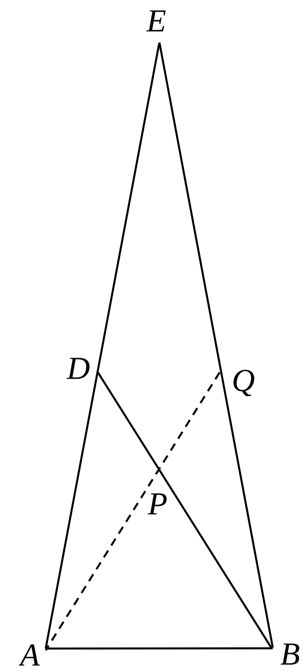
Extend AP to meet BE at Q.

(a) Find $\angle QAB$.



(b) Using (a), prove that $\Delta DAB \cong \Delta QBA$.

(c) Name the corresponding side of *DB*.



(d) Using (c), prove that $PD = PQ$.

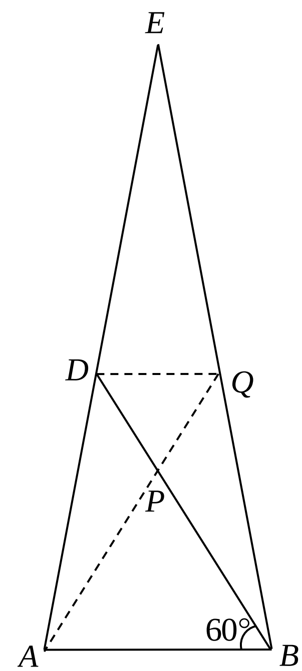
F.2 Math CL Geometry (Expert Summary Group 4)

Let P be a point on DB so that ΔAPB is an equilateral triangle. (**from Expert Group 1**)

Extend AP to meet BE at Q.

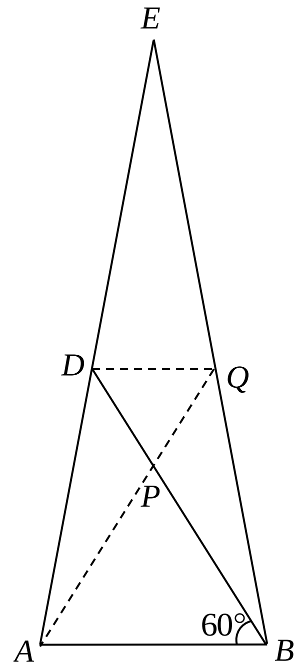
It is given that $PD = PQ$. (**proved by Expert Group 3**)

(a) Find $\angle DPQ$.



(b) Using (a), prove that ΔDPQ is an equilateral triangle.

(c) Using (b), prove that $DQ // AB$.



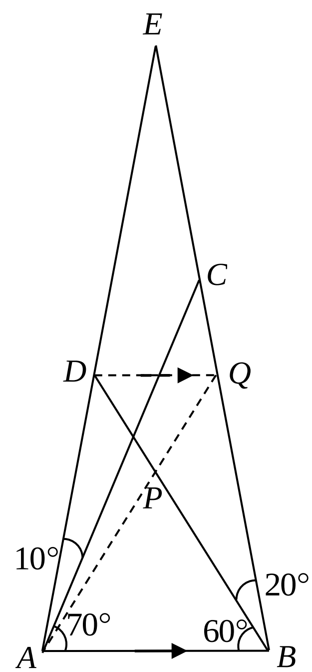
F.2 Math CL Geometry (Expert Summary Group 5)

Let P be a point on DB so that ΔAPB is an equilateral triangle. (**from Expert Group 1**)

Extend AP to meet BE at Q.

It is given that $DQ \mathbin{//} AB$. (**proved by Expert Group 4**)

(a) Find $\angle EDQ$ and $\angle EQD$.



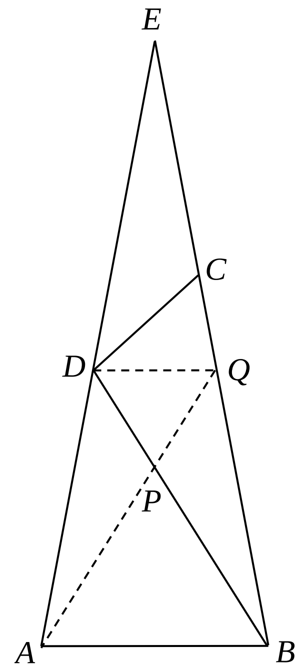
(b) Using (a), prove that $ED = EQ$.

It is also given that

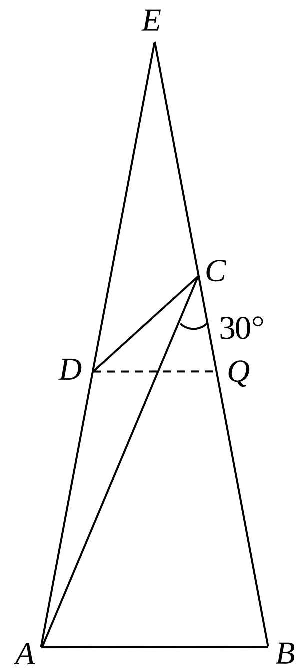
1. $DB = DE$ (**proved by Expert Group 2**)

2. $BP = EC = x$ (**proved by Expert Group 2**)

3. $DP = PQ = QD = y$ (**proved by Expert Group 4**)



(c) Using (b), prove that $QC = QD = y$.



(d) Using (a) and (c), find $\angle QCD$ and $\angle ACD$.