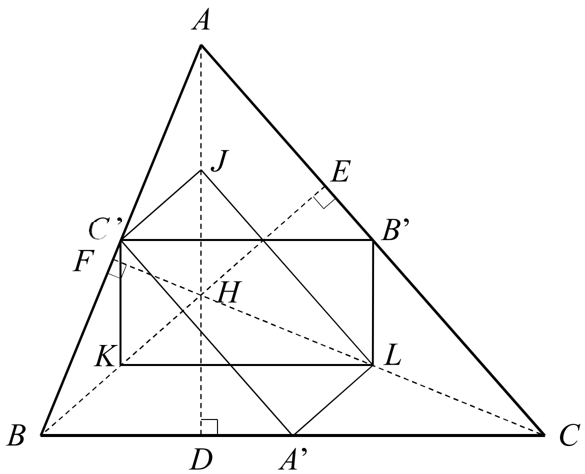
Exercise on Feuerbach Circle (2022 version)

1. Prove that the circumcentre of a right-angled triangle is the mid-point of its hypotenuse.

2. Let A', B' and C' be the mid-points of BC, CA and AB respectively in ΔABC. D, E and F are the feet of the altitudes from A, B and C to BC, CA and AB respectively.
   (a) Prove that $\angle B'A'C' = \angle C'AB'$.
   (b) Prove that $\angle B'DC' = \angle B'AC'$.
   (c) Prove that A', B', C', D, E and F are concyclic.

3. ΔABC is an acute-angled triangle. D, E and F are points lying on BC, CA and AB respectively such that AD, BE and CF are the altitudes of ΔABC. H is the orthocentre of ΔABC. A', B' and C' are the mid-points of BC, CA and AD respectively. Let $\angle ABH = x$.
   (a) Prove that $FBDH$ and $BFEC$ are cyclic quadrilaterals.
   (b) Express $\angle HDF$ and $\angle HDE$ in terms of x.
   (c) Prove that A' is the circumcentre of $BFEC$.
   (d) Express $\angle EA'F$ in terms of x.
   (e) Prove that A', B', C', D, E and F are concyclic.

4. In the figure, A', B' and C' are the mid-points of BC, CA and AB respectively. D, E and F are points lying on BC, CA and AB respectively such that AD, BE and CF are the altitudes of ΔABC. H is the orthocentre of ΔABC. J, K and L are the mid-points of AH, BH and CH respectively.



   (a) Prove that $B'C'KL$ and $C'A'LJ$ are rectangles and, hence, prove that $A'J$, $B'K$ and $C'L$ intersect at the same point.
   (b) Prove that J, C', D and A' are concyclic.
   (c) Prove that A', B', C', D, E, F, J, K and L are concyclic and

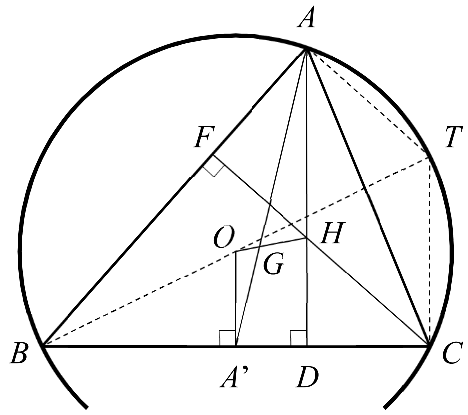
the centre of the circle passing through these 9 points is the midpoint of $C'L$.

DEFINITION The circumcircle of $\Delta A'B'C'$ is defined as the *nine-point circle* (or the *Feuerbach Circle*) of ΔABC where A' , B' and C' are the mid-points of BC , CA and AB respectively.

REMARKS: 1. The above questions tell us that the circumcircles of $\Delta A'B'C'$, ΔDEF and ΔJKL are actually the same circle.

2. The main objective of this exercise is to give two proofs to the fact that the nine-point circle of a triangle is always tangent to its incircle.

5. ΔABC is an acute-angled triangle. O is the circumcentre of ΔABC . A' is the mid-point of BC . AD and CF are the altitudes. H is the orthocentre. G is the centroid. Let T be a point on the circumcircle such that BP forms a diameter of that circle.



(a) Prove that $AHCT$ is a parallelogram.
Hence, prove that $AH = 2\ OA'$.

(b) Prove that $\Delta AGH \sim \Delta A'GO$.
Hence, prove that O , G and H are collinear and $HG : GO = 2 : 1$.
(OGH is called the *Euler line* of ΔABC .)

(c) Let J and N be the midpoints of AH and OH respectively.

(i) Prove that $JOA'H$ is a parallelogram.

(ii) Prove that N is the centre of the nine-point circle of ΔABC .

(iii) Prove that $OG : GN : NH = 2 : 1 : 3$.

(iv) Prove that $\Delta HAO \sim \Delta HJN$.
Hence, prove that $OA = 2\ NJ$, i.e. the radius of the circumcircle of ΔABC is twice the radius of its nine-point circle.

(v) Prove that $AOA'J$ is also a parallelogram.

(d) Let P be any point on the circumcircle ABC and Q be the mid-point of HP . Prove that Q lies on the nine-point circle.

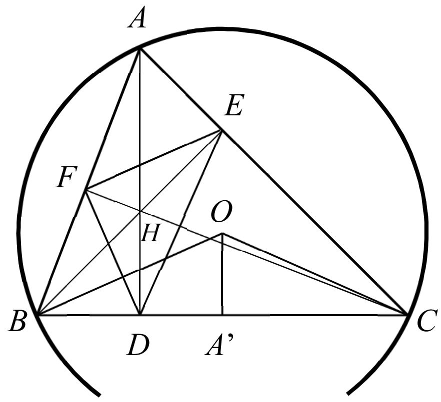
6. ΔABC is an acute-angled triangle. H is the orthocentre of ΔABC .

(a) Locate the foot of the altitude from each of the vertices of ΔABH to its opposite side and hence, locate the orthocentre of ΔABH .

(b) Locate the nine-point circle of ΔABH .

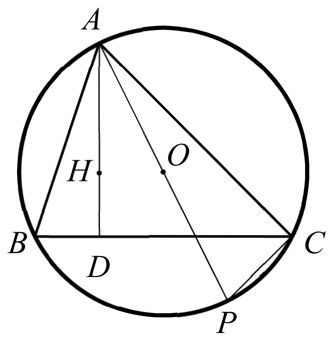
(c) Prove that the radius of the circumcircle of ΔABH is equal to the radius of the circumcircle of ΔABC .

7. In the figure, ΔABC is an acute-angled triangle. O is the circumcentre. H is the orthocentre. D, E and F are the feet of the altitudes on BC, CA and AB respectively. A' is the mid-point of AB.



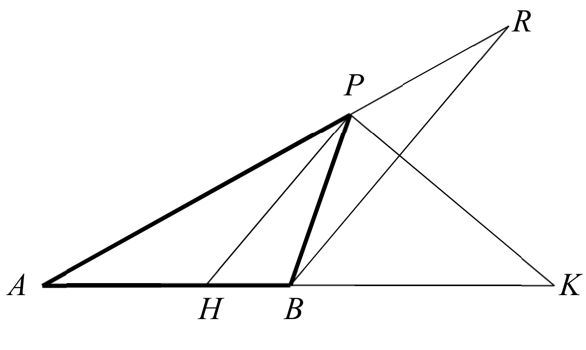
   (a) Prove that $\angle BAC = \angle BOA'$.
   Hence, prove that $\angle OBC = \angle HBA$.
   (b) Prove that $OB \perp FD$.
   (c) Prove that $\angle HAO = |\angle ABC - \angle ACB|$.
   (d) Suppose W is a point lying on the nine-point circle of ΔABC.
   Express $\angle A'WD$ in terms of $\angle ABC$ and $\angle ACB$.
   (e) If AD is produced to meet the circumcircle at Q, prove that $HQ = 2HD$.

8. In the figure, ΔABC is an acute-angled triangle. O is the circumcentre. H is the orthocentre. D is the feet of the altitude on BC. P is a point on the circumcircle of ΔABC such that AP passing through O.



   (a) Prove that $\angle OAC = \angle HAB$.
   (b) IA bisects $\angle HAO$ where I is the incentre of ΔABC.

9. (a) In the figure, H is a point lying on AB. BR is a line passing through B parallel to PH intersecting AP produced at R.



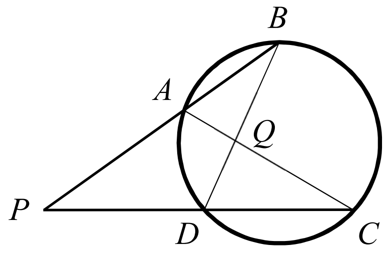
   (i) Prove that ΔBPR is isosceles and $\dfrac{PA}{PB} = \dfrac{AH}{BH}$ if PH is the angle bisector of $\angle APB$.
   (ii) Prove that PH is the angle bisector of $\angle APB$ if $PA : PB = AH : BH$.

   (b) In the same figure, K is a point on AB produced.
   (i) Prove that $\dfrac{PA}{PB} = \dfrac{AK}{BK}$ if PK is the angle bisector of $\angle BPR$.
   (ii) Prove that PK is the angle bisector of $\angle BPR$ if $PA : PB = AK : BK$.

   (c) A and B are two fixed points and let k be a fixed positive real number not equal to 1. P is a variable point such that $\dfrac{PA}{PB} = k$.
   Describe the *locus* of P. Also describe what would happen if k equals 1.

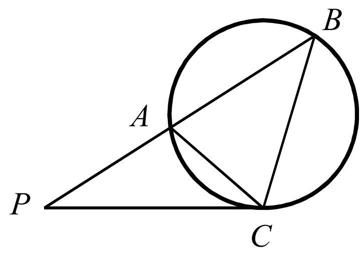
10. Let O, G, H and I be the circumcentre, centroid, orthocentre and incentre of ΔABC respectively.
    (a) Prove that O, G, H and I are collinear if ΔABC is isosceles.
    (b)* Prove that ΔABC is isosceles if O, G, H and I are collinear.

11. (a) In the figure, A, B, C and D are concyclic points. P is an external point outside the circle such that PAB and PDC are straight lines. Q is the intersection point of AC and BD.

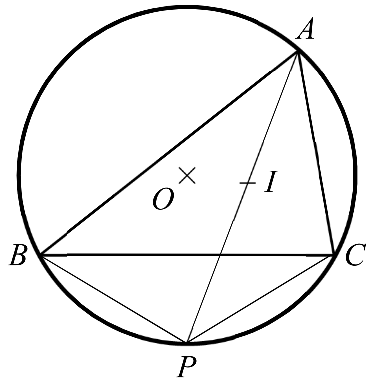


    (i) Prove that $\Delta PAC \sim \Delta PDB$ and $PA \cdot PB = PC \cdot PD$.
    (ii) Prove that $\Delta QAB \sim \Delta QDC$ and $QA \cdot QC = QB \cdot QD$.
    (b) Conversely, if PAB and PDC form two straight lines such that $PA \cdot PB = PC \cdot PD$, prove that A, B, C and D are concyclic points.

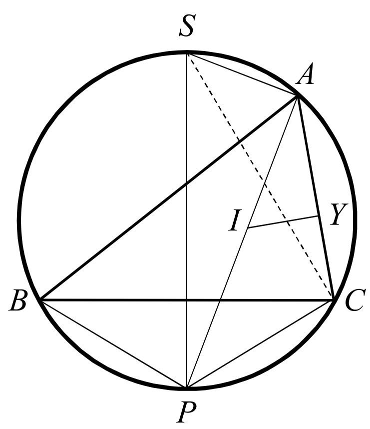
12. (a) In the figure, PC is tangent to the circle ABC at C. Prove that $PC^2 = PA \cdot PB$.
    (b) Conversely, if PAB forms a straight line and C is another point such that $PC^2 = PA \cdot PB$, prove that PC is tangent to the circle ABC at C.



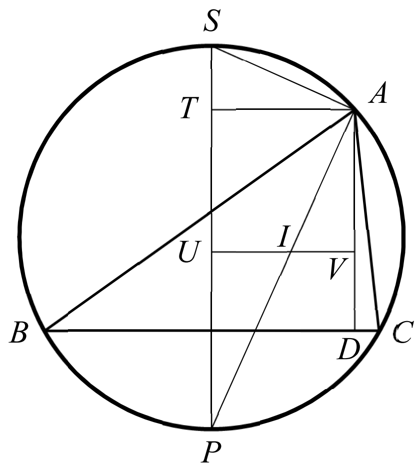
13. In the figure, it shows a circle passing through A, B and C. I is the incentre of ΔABC and AI is produced to meet the circle at P.
    (a) Prove that $PB = PI = PC$.
    (b) Suppose O is the centre of the circle in the figure. What is the relation between OP and BC? What conclusion can you draw from this observation? Explain your answer.



14. In the figure, SP is the diameter of the circumcircle of ΔABC such that $SP \perp BC$. Y is a point on AC such that $IY \perp AC$ where I is the incentre of ΔABC.
    (a) Prove that $\Delta IAY \sim \Delta PSC$.
    (b) Deduce that $AI \cdot IP = 2Rr$ where R and r are the radii of the circumcircle and incircle respectively.
    (c) Let O be the circumcentre of ΔABC. Prove that $IO^2 = R^2 - 2Rr$.



15. In the figure, SP is the diameter of the circumcircle of ΔABC such that $SP \perp BC$. T is a point on SP such that $AT \parallel BC$. D is a point on BC such that AD is the altitude from A to BC. U and V are points on SP and AD respectively such that $UV \parallel BC$ and passing through I where I is the incentre of ΔABC.

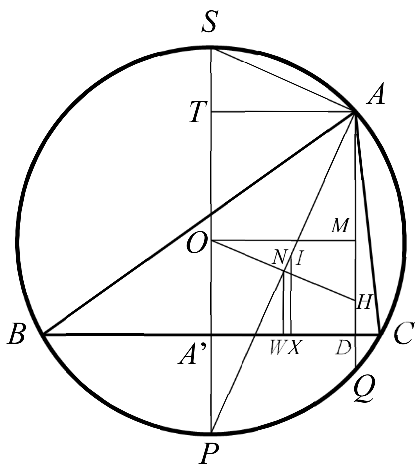


(a) Prove that $\dfrac{UI}{IP} = \dfrac{ST}{AS}$.

(b) Prove that $\dfrac{IV}{AI} = \dfrac{AS}{SP}$.

(c) Using (a), (b) and a result from the previous question, prove that $UI \cdot IV = r \cdot ST$ where r is the radius of the incircle.

16. In the figure, SP is the diameter of the circumcircle of ΔABC such that $SP \perp BC$. T is a point on SP such that $AT \parallel BC$. D is a point on BC such that AD is the altitude from A to BC. AD is produced to meet the circumcircle at Q. O, H and I are the circumcentre, orthocentre and incentre of ΔABC respectively. A', M and N are the midpoints of BC, AQ and OH respectively. W and X are points on BC such that $NW \perp BC$ and $IX \perp BC$. The radii of the circumcircle and incircle of ΔABC are denote by R and r respectively.

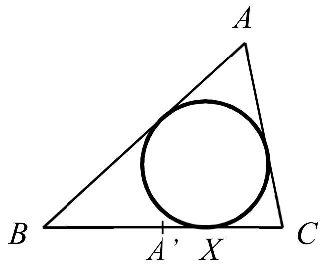


(a) Using the fact that $NW = ½\,(OA' + HD)$, prove that $NW = ½\,AM$.

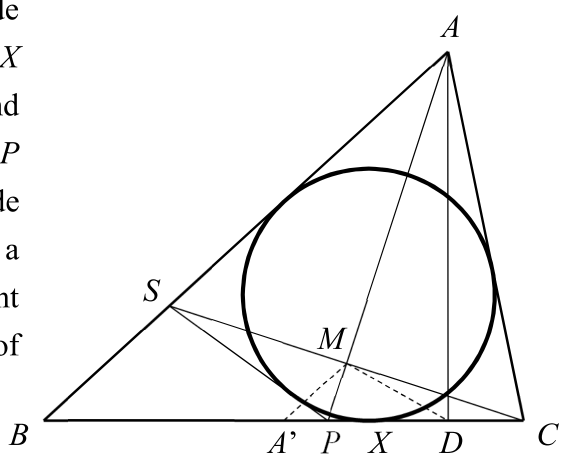
(b) Prove that $A'X \cdot XD = A'W^2 - WX^2$ and, hence, $WX^2 = ¼\,OM^2 - r \cdot ST$.

(c) Using the fact that $IN^2 = (IX - NW)^2 + WX^2$, prove that $IN = ½R - r$.
What relation between the nine-point circle and the incircle of ΔABC can be deduced from the above result? Explain your answer.

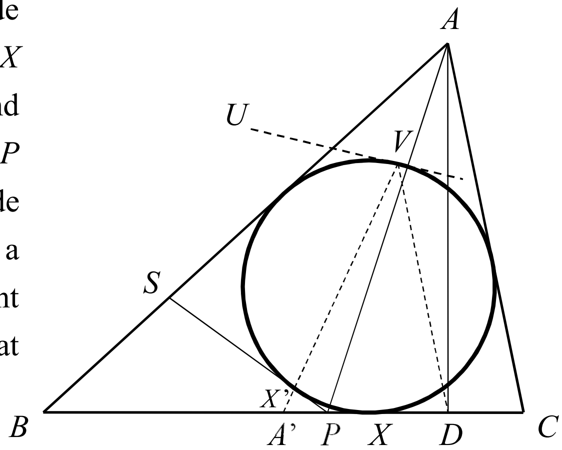
17. The figure shows a circle inscribed inside ΔABC. If the circle touches BC at X and A' is the mid-point of BC, prove that $A'X = ½\,|\,AB - AC\,|$.



18. The figure shows a circle inscribed inside ΔABC. The circle touches BC at X and A' is the mid-point of BC. P and D are points on BC such that AP bisects $\angle BAC$ and AD is the altitude from A to BC respectively. S is a point on AB such that PS is also tangent to the incircle. M is the intersection point of AP and SC. Prove that
(a) $AS = AC$,
(b) $A'M \,//\, BA$,
(c) $A'M = ½ (AB - AC)$,
(d) A, M, D and C are concyclic,
(e) $\angle A'MP = \angle MDP$ and
(f) $A'X^2 = A'P \cdot A'D$.



19. The figure shows a circle inscribed inside ΔABC. The circle touches BC at X and A' is the mid-point of BC. P and D are points on BC such that AP bisects $\angle BAC$ and AD is the altitude from A to BC respectively. S is a point on AB such that PS is also tangent to the incircle. Let PS touch the incircle at X'. $A'X'$ is produced to meet the incircle again V.



(a) Prove that V, X', P and D are concyclic.
(b) Using the fact that $\angle X'VD = \angle X'PA'$, express $\angle A'VD$ in terms of $\angle ABC$ and $\angle ACB$.
Hence, prove that V lies on the nine-point circle of ΔABC.
(c) Let UV be a line tangent to the incircle at V as shown in the figure.
Using the fact that $\angle UVX' = \angle SX'V$, prove that UV is also tangent to the nine-point circle of ΔABC at V.
What relation between the nine-point circle and the incircle of ΔABC can be deduced from the above result? Explain your answer.

HINT to Question 10(b): It suffices to show the straight line OIH must pass through one of the vertices of the given triangle. If not, it is possible to prove that the three vertices of the given triangle are concyclic with I but this result is absurd.

Suggested Solutions

1. Let O be the circumcentre of ΔABC such that $\angle ABC = 90°$.
   Then $\angle AOC = 2 \times 90°$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$)
   $= 180°$
   Hence, AOC forms a straight line and it is the same as the hypotenuse AC.
   $\because$ $OA = OC$,
   $\therefore$ the circumcentre O is the mid-point of the hypotenuse AC.

2. (a) $\because$ $A'C' \mathbin{//} AC$ and $A'B' \mathbin{//} AB$, (mid-point theorem)
   $\therefore$ $A'B'AC'$ is a parallelogram.
   Hence, $\angle B'A'C' = \angle C'AB'$ (opp. $\angle$s of //gram)

   (b) $\because$ B' is the mid-point of AC, which is the hypotenuse of ΔADC,
   $\therefore$ $AB' = DB'$ (result in Q1)
   Similarly, $AC' = DC'$.
   $\because$ $AD = AD$ (common)
   $\therefore$ $\Delta AC'B' \cong \Delta DC'B'$ (S.S.S.)
   Hence, $\angle B'DC' = \angle B'AC'$ (corr. $\angle$s of cong. Δs)

   (c) Now, $\angle B'A'C' = \angle B'DC'$.
   $\therefore$ B', D, A' and C' are concyclic. (converse of $\angle$s in the same seg.)
   Similarly, we can prove A', F, C' and B' are concyclic and C', E, B' and A' are concyclic.
   As a conclusion, A', B', C', D, E and F are concyclic.

3. (a) $\because$ $\angle BFH + \angle BDH = 180°$
   $\therefore$ $FBDH$ is a cyclic quadrilateral.
   $\because$ $\angle BFC = 90° = \angle BEC$,
   $\therefore$ $BFEC$ is a cyclic quadrilateral.

   (b) $\angle HDF = x$ ($\angle$s in the same segment)
   Similarly, $ECDH$ is also a cyclic quadrilateral.
   $\therefore$ $\angle HDE = \angle HCE$ ($\angle$s in the same segment)
   $= \angle HDB$ ($\angle$s in the same segment)
   $= x$

   (c) By the result in Q.1, A' is the circumcentre of $BFEC$.

   (d) $\angle EA'F = 2x$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$)

   (e) $= \angle EDF$ (results in (b))
   $\therefore$ E, D, A' and F are concyclic. (converse of $\angle$s in the same seg.)
   Similarly, A', B', C', D, E and F are concyclic.

4. (a) $C'K \parallel AH$ and $B'L \parallel AH$ (mid-point theorem)

Hence, $C'K \parallel B'L$.

Similarly, $C'B' \parallel BC \parallel KL$.

$\therefore$ $B'C'KL$ is a parallelogram. (definition)

Note that $AH \perp BC$. Hence, $B'C' \perp C'K$.

$\therefore$ $B'C'KL$ is a rectangle. (definition)

Similarly, $C'A'LJ$ is a rectangle, too.

$C'L$ intersects $B'K$ at the mid-point of $C'L$. (property of rectangle)

Similarly, $C'L$ intersects $A'J$ intersect at the mid-point of $C'L$.

Therefore, $A'J$, $B'K$ and $C'L$ intersect at the same point.

(b) $\because$ $JC'A' = 90°$ (definition of rectangle)

$= JDA'$ (definition of altitude)

$\therefore$ J, C', D and A' are concyclic. (converse of $\angle$s in the same seg.)

(c) Let N be the mid-point of $C'L$.

From (a), we know that $NA' = NB' = NC' = NJ = NK = NL$. (prop. of rectangle)

Hence, A', B', C', J, K and L are concyclic.

From (b), we also know that D, E and F also lie on the same circle with N as the centre.

5. (a) As BT is a diameter, $\angle BCT = \angle BAT = 90°$ ($\angle$ in semi-circle).

Hence, $AD \parallel TC$ and $AT \parallel FC$ (corr. $\angle$s equal),

i.e. $AHCT$ is a parallelogram and $AH = TC$.

$\because$ A' is the mid-point of BC,

$\therefore$ $OA' : AH = OA' : TC = BA' : BC = 1 : 2$, i.e. $AH = 2\,OA'$.

(b) $A'G : AH = 1 : 2$ (property of centroid)

$\angle OA'G = \angle HAG$ (alt. $\angle$s, $OA' \parallel AD$)

$\therefore$ $\Delta OA'G \sim \Delta HAG$ (ratio of 2 sides, inclined $\angle$)

Hence $\angle A'GO = \angle AGH$ (corr. $\angle$s of similar Δs)

i.e. O, G and H are collinear (converse of vert. opp. $\angle$s).

$HG : GO = 1 : 2$ which is a consequence of $\Delta OA'G \sim \Delta HAG$.

(c) (i) $\because$ $AH = 2\,JH$,

$\therefore$ $JH = OA'$ and $JH \parallel OA'$.

Hence, $JOA'H$ is a parallelogram. (opp. sides equal and //)

(ii) $\because$ N is the mid-point of OH,

$\therefore$ it is also the mid-point of $A'J$. (diags. of //gram)

By the result in 4(c), N is the centre of the nine-point circle.

(iii) $\because$ $OG : OH = 1 : 2 = 2 : 4$ and $ON : NH = 1 : 1 = 3 : 3$,

$\therefore \quad OG : GN : NH = 2 : 1 : 3$.

(iv) $\because \quad \dfrac{HN}{HO} = \dfrac{HJ}{HA} = \dfrac{1}{2}$ and $\angle OHA = \angle NHJ$

$\therefore \quad \Delta HAO \sim \Delta HJN$ (ratio of 2 sides, inclined $\angle$)

Hence, $OA = 2\,NJ$ (corr. sides of similar Δs)

(v) $\because \quad AJ = OA'$ and $AJ \,//\, OA'$,

$\therefore \quad AOA'J$ is a parallelogram. (opp. sides equal and //)

(d) Consider ΔPOH and ΔQNH .

$PH : QH = 2 : 1$ (by definition)

$OH : NH = 2 : 1$ (by definition)

$\angle PHO = \angle QHN$ (common)

$\therefore \quad \Delta POH \sim \Delta QNH$ (ratio of 2 sides, inclined $\angle$)

Hence, $NQ = \dfrac{1}{2} OP$ (corr. sides of similar Δs)

Note that OP is the radius of the circumcircle, NQ equals the radius of the nine-point circle, by (c)(iv), and N is the centre of the nine-point circle, by (c)(ii).

$\therefore \quad Q$ is a point on the nine-point circle.

6. (a) The feet of the altitudes from A , B and H to their opposite sides in ΔABH are E , D and F respectively. The orthocentre is C .

(b) Since the nine-point circle is the circle passing through the feet of altitudes, the nine-point circle of ΔABH is the same as that of ΔABC .

(c) Since the two triangles share the same nine-point circle and the radius of the nine-point circle is half of that of the circumcircle (result in 5(c)(iv)), the radius of the circumcircle of ΔABH is equal to that of the circumcircle of ΔABC .

7. (a) $\angle BAC = 2\,\angle BOC$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$)

$= \angle BOA'$ (prop. of isosceles Δ)

Hence, $\angle OBC = 90° - \angle BOA'$ ($\angle$ sum of Δ)

$= 90° - \angle BAC$ (proved)

$= \angle HBA$ ($\angle$ sum of Δ)

(b) $\because \quad B$, D , H and F are concyclic, (opp. angles supp.)

$\therefore \quad \angle FDB = \angle FHB$ ($\angle$s in the same segment)

$= 90° - \angle HBA$ ($\angle$ sum of Δ)

$= 90° - \angle OBC$ (result in (a))

Let FD intersect OB at X .

Then $\angle BXD = 180° - \angle OBC - \angle FDB$ ($\angle$ sum of Δ)

$= 90°$

$\therefore \quad OB \perp FD$.

(c) $\angle HAO = \angle DAC - \angle OAC$

$= \angle DAC - \angle DAB$ (result in (a))

$= (90^\circ - \angle ACB) - (90^\circ - \angle ABC)$

$= \angle ABC - \angle ACB$

As it is possible that $\angle ABC < \angle ACB$, an absolute sign is needed in the above result. i.e. $\angle HAO = |\angle ABC - \angle ACB|$

(d) First, assume W is lying on the same side of A with respect to BC.

By the result in 5(c)(v), $AOA'J$ is a parallelogram. $JA' \mathbin{//} AO$.

$\therefore \quad \angle A'WD = \angle A'JD$ ($\angle$s in the same segment)

$= \angle DAO$ (corr. $\angle$s, $JA' \mathbin{//} AO$)

$= |\angle ABC - \angle ACB|$ (result in (c))

If W is lying on the opposite side of A, we have

$\angle A'WD = 180^\circ - |\angle ABC - \angle ACB|$.

(e) $\angle HBD = 90^\circ - \angle ACB$ ($\angle$ sum of Δ)

$= 90^\circ - \angle AQB$ ($\angle$s in the same segment)

$= \angle QBD$ ($\angle$ sum of Δ)

$BD = BD$ (common)

$\angle BDH = \angle BDQ$ (adj. $\angle$s on st. line)

$\therefore \quad \Delta BHD \cong \Delta BQD$ (A.S.A.)

Hence, $HD = QD$ (corr. sides of cong. Δs)

i.e. $HQ = 2HD$.

8. (a) (This is the same as the result in Question 7(a).)

$\angle APC = \angle ABC$ ($\angle$s in same segment)

$\angle ACP = 90^\circ$ (definition of altitude)

$= \angle ADB$ ($\angle$ in semi-circle)

$\therefore \quad \angle OAC = \angle HAB$ ($\angle$ sum of Δ)

(b) By definition, IA bisects $\angle BAC$.

Since $\angle OAC = \angle HAB$, I also bisects $\angle HAO$.

9. (a) (i) $\angle PBR = \angle BPH$ (alt. $\angle$s, $PH \mathbin{//} BR$)

$= \angle APH$ (given)

$= \angle PRB$ (corr. $\angle$s, $PH \mathbin{//} BR$)

$\therefore \quad PB = PR$ (sides opp. equal $\angle$s)

$\dfrac{PA}{PB} = \dfrac{PA}{PR} = \dfrac{AH}{BH}$ (intercept theorem)

(ii) $\because \quad \dfrac{PA}{PB} = \dfrac{AH}{BH}$ (given) and $\dfrac{PA}{PR} = \dfrac{AH}{BH}$ (intercept theorem),

$\therefore \quad PR = PB$.

Hence, $\angle APH = \angle PRB$ (corr. $\angle$s, $PH // BR$)

$= \angle PBR$ (base $\angle$s of isos. Δ)

$= \angle BPH$ (alt. $\angle$s, $PH // BR$)

$\therefore \quad PH$ bisects $\angle APB$.

(b) (i) Let Q be a point on AP such that $QB // PK$.

$\angle PBQ = \angle BPK$ (alt. $\angle$s, $QB // PK$)

$= \angle RPK$ (given)

$= \angle PQB$ (corr. $\angle$s, $QB // PK$)

$\therefore \quad PB = PQ$ (sides opp. equal $\angle$s)

$\dfrac{PA}{PB} = \dfrac{PA}{PQ} = \dfrac{AK}{BK}$ (intercept theorem)

(ii) $\because \quad \dfrac{PA}{PB} = \dfrac{AK}{BK}$ (given) and $\dfrac{PA}{PQ} = \dfrac{AK}{BK}$ (intercept theorem),

$\therefore \quad PQ = PB$.

Hence, $\angle BPK = \angle PBQ$ (corr. $\angle$s, $PH // BR$)

$= \angle PQB$ (base $\angle$s of isos. Δ)

$= \angle RPK$ (alt. $\angle$s, $PH // BR$)

$\therefore \quad PK$ bisects $\angle BPR$.

(c) If $k \neq 1$, K is well-defined. By definition, $\angle HPK = 90°$.

$\therefore$ the locus of P is a circle using HK as its diameter.

If $k = 1$, K does not exist. PH will be the perpendicular bisector of AB .

10. (a) Assume $AB = AC$ if ΔABC is isosceles.

Let D be the mid-point of BC . Then it is easy to prove that $AD \perp BC$ and AD bisects $\angle BAC$. Therefore, AD is the perpendicular bisector, the median, the altitude and the angle bisector of the triangle.

Hence, O , G , H and I are lying on AD . They are collinear.

(b) As O , G and H must always be collinear, it is sufficient to explain why ΔABC is isosceles if O , I and H are collinear.

It is also easy to show ΔABC to be isosceles if the straight line OIH passes through one of the vertices of the triangle. Therefore, it is sufficient to prove that the straight line OIH must pass through one of the vertices of ΔABC if O , I and H are collinear.

Here, we assume that the straight line OIH does *not* pass through *any* vertices of ΔABC . As O , I , H and A do not form a straight line, by the result in

8(b), IA must bisect $\angle HAO$. Then by the result in 9(a), $\frac{AH}{AO} = \frac{IH}{IO}$.
By the same argument, we have $\frac{BH}{BO} = \frac{CH}{CO} = \frac{IH}{IO}$. By the result in 9(c), we can conclude that A, B, C and I are concyclic.
However, the above result is absurd as the incentre of a triangle cannot lie on its circumcircle. Therefore, it is impossible to have the straight line OIH not passing through any vertices of ΔABC.

11. (a) (i)

	$\angle PCA = \angle PBD$	($\angle$s in the same segment)
	$\angle APC = \angle DPB$	(common)
	$\angle PAC = \angle PDB$	($\angle$ sum of Δ)
$\therefore$	$\Delta PAC \sim \Delta PDB$	(A.A.A.)
Hence,	$\frac{PA}{PD} = \frac{PC}{PB}$	(corr. sides of similar Δs)
i.e.	$PA \cdot PB = PC \cdot PD$.	

(ii)

	$\angle QAB = \angle QDC$	($\angle$s in the same segment)
	$\angle AQB = \angle DQC$	(vert. opp. $\angle$s)
	$\angle QBA = \angle QCD$	($\angle$s in the same segment)
$\therefore$	$\Delta QAB \sim \Delta QDC$	(A.A.A.)
Hence,	$\frac{QA}{QB} = \frac{QD}{QC}$	(corr. sides of similar Δs)
i.e.	$QA \cdot QC = QB \cdot QD$.	

(b)

$\because$	$PA \cdot PB = PC \cdot PD$	(given)
$\therefore$	$\frac{PA}{PD} = \frac{PC}{PB}$	
Since	$\angle APC = \angle DPB$	(common),
	$\Delta PAC \sim \Delta PDB$	(ratio of 2 sides, inclined $\angle$)
$\therefore$	$\angle PCA = \angle PBD$	(corr. $\angle$s of similar Δs)
Hence,	A, B, C and D are concyclic.	(converse of $\angle$s in the same seg.)

12. (a)

	$\angle PCA = \angle PBC$	($\angle$s in alt. segment)
	$\angle APC = \angle CPB$	(common)
	$\angle PAC = \angle PCB$	($\angle$ sum of Δ)
$\therefore$	$\Delta PAC \sim \Delta PCB$	(A.A.A.)
Hence,	$\frac{PC}{PB} = \frac{PA}{PC}$	(corr. sides of similar Δs)
i.e.	$PC^2 = PA \cdot PB$.	

(b)

$\because$	$PC^2 = PA \cdot PB$	(given)

$\therefore \quad \dfrac{PC}{PB} = \dfrac{PA}{PC}$

Since $\angle APC = \angle CPB$ (common),

$\Delta PAC \sim \Delta PCB$ (ratio of 2 sides, inclined $\angle$)

$\therefore \quad \angle PCA = \angle PBC$ (corr. $\angle$s of similar Δs)

Hence, PC is tangent to the circle ABC at C. (converse of $\angle$s in alt. segment)

13. (a) Let $\angle A = 2a$ and $\angle B = 2b$.
Then $\angle IBP = a + b$ ($\angle$s in same seg.) while $\angle BIP = a + b$ (ext. $\angle$ of Δ).
Therefore $PB = PI$ (sides opp. equal $\angle$s). Similarly, $PI = PC$.

(b) Note that $OB = OC$ (radius) and $PB = PC$ (proved). Hence, $OBPC$ is a kite and OP is the perpendicular bisector of BC.
Therefore, *the angle bisector of an interior angle of a triangle intersects the perpendicular bisector of its opposite side at a point on the circumcircle*.

14. (a)

$\angle IAY = \angle PSC$ ($\angle$s in the same segment)

$\angle IYA = 90° = \angle PCS$ ($\angle$ in semi-circle)

$\angle AIY = \angle SPC$ ($\angle$ sum of Δ)

$\therefore \quad \Delta IAY \sim \Delta PSC$ (A.A.A.)

(b) Hence, $\dfrac{IA}{PS} = \dfrac{IY}{PC}$ (corr. sides of similar Δs)

Since $PS = 2R$, $IY = r$ and $PC = IP$,

we have $\dfrac{IA}{2R} = \dfrac{r}{IP}$

i.e. $AI \cdot IP = 2Rr$.

(c) Produce IO to meet the circumcircle ABC. Then this line segment will become a diameter of the circle and it is cut by AP at I into two parts with lengths $R + IO$ and $R - IO$. By the result in 11(a)(ii), we have

$$(R + IO) \times (R - IO) = AI \cdot IP = 2Rr.$$

i.e. $R^2 - IO^2 = 2Rr$

or $IO^2 = R^2 - 2Rr$.

15. (a)

$\angle IPU = 90° - \angle TSA$ ($\angle$ sum of Δ) ($\angle$ in semi-circle)

$= \angle SAT$ ($\angle$ sum of Δ)

$\angle IUP = 90° = \angle STA$ (given)

$\angle UIP = \angle TSA$ ($\angle$ sum of Δ)

$\therefore \quad \Delta UIP \sim \Delta TSA$ (A.A.A.)

Hence, $\dfrac{UI}{IP} = \dfrac{ST}{AS}$ (corr. sides of similar Δs)

(b)
$$\angle VAI = \angle APS \quad \text{(alt. } \angle\text{s, } SP \,//\, AD)$$
$$\angle AVI = 90° = \angle PAS \quad \text{(given)}$$
$$\angle AIV = \angle PSA \quad (\angle \text{ sum of } \Delta)$$
$$\therefore \quad \Delta VIA \sim \Delta ASP \quad \text{(A.A.A.)}$$
Hence,
$$\frac{IV}{AI} = \frac{AS}{SP} \quad \text{(corr. sides of similar } \Delta\text{s)}$$

(c) Then
$$\frac{UI}{IP} \times \frac{IV}{AI} = \frac{ST}{AS} \times \frac{AS}{SP}$$
$$UI \cdot IV = \frac{IP \times AI}{SP} \cdot ST$$
$$= \frac{2Rr}{2R} \cdot ST \quad \text{(result in 14(b)) (definition)}$$
$$= r \cdot ST .$$

16. (a) $\because$ $HD \,//\, NW \,//\, OA'$ and N is the mid-point of OH,
$$\therefore \quad NW = \tfrac{1}{2}(OA' + HD)$$
$$= \tfrac{1}{2}(MD + DQ) \quad \text{(result in 7(e))}$$
$$= \tfrac{1}{2} AM .$$

(b)
$$A'X \cdot XD = (A'W + WX)(WD - WX)$$
$$= (A'W + WX)(A'W - WX)$$
$$= A'W^2 - WX^2$$
Therefore,
$$WX^2 = A'W^2 - A'X \cdot XD$$
$$= (\tfrac{1}{2} OM)^2 - UI \cdot IV \quad \text{(notation as in Q.15)}$$
$$= \tfrac{1}{4} OM^2 - r \cdot ST .$$

(c)
$$IN^2 = (IX - NW)^2 + WX^2 \quad \text{(Pythagoras Theorem)}$$
$$= (r - \tfrac{1}{2} AM)^2 + \tfrac{1}{4} OM^2 - r \cdot ST$$
$$= r^2 - r \cdot AM + \tfrac{1}{4} AM^2 + \tfrac{1}{4} OM^2 - r \cdot ST$$
$$= r^2 - r \cdot R + \tfrac{1}{4} R^2 \quad (AM + ST = R,\ AM^2 + OM^2 = R^2)$$
$$= (\tfrac{1}{2} R - r)^2$$
$$\therefore \quad IN = \tfrac{1}{2} R - r .$$

Note that N is the centre of the nine-point circle of ΔABC. $\tfrac{1}{2} R$ equals the radius of the nine-point circle. Therefore, "$IN = \tfrac{1}{2} R - r$" means the distance between the centre of the nine-point circle and that of the incircle is equal to the difference between the radius of the nine-point circle and that of the incircle. In other words, the nine-point circle always intersect the incircle at one and only one point. That is, the nine-point circle of a triangle is always tangent to its incircle.

17. Let the circle touch AC and AB at Y and Z respectively.
Assume $AB > AC$. Then A' is closer to B than X.

$\therefore$ $A'X = BX - BA'$
$= BZ - CA'$
$= (AB - AZ) - (CX + A'X)$
$= AB - AY - CX - A'X$
$= AB - (AC - CY) - CY - A'X$
$= AB - AC - A'X$

i.e. $A'X = \frac{1}{2}(AB - AC)$

If $AB < AC$, then $A'X = \frac{1}{2}(AC - AB)$

If $AB = AC$, then $A'X = 0$

As a conclusion, $A'X = \frac{1}{2}\,|\,AB - AC\,|$.

18. (a) $\because$ AP passes through the centre of the incircle, i.e. AP bisects $\angle BAC$,

$\therefore$	$\angle SAP = \angle CAP$	(definition)
	$AP = AP$	(common)
and	$\angle SPA = \angle CPA$	(tangent properties)
$\therefore$	$\Delta APS \cong \Delta APC$	(A.S.A.)
Hence,	$AS = AC$	(corr. sides of cong. Δs)

(b)

	$\angle SAM = \angle CAM$	(given)
	$AM = AM$	(common)
$\therefore$	$\Delta ASM \cong \Delta ACM$	(S.A.S.)
Hence,	$SM = CM$	(corr. sides of cong. Δs)

i.e. M is the mid-point of CS.

$\because$ A' is mid-point of CB,		(given)
$\therefore$	$A'M \parallel BS$ (i.e. BA)	(mid-point theorem)

(c)

and	$A'M = \frac{1}{2}\,BS$	(mid-point theorem)
	$= \frac{1}{2}\,(AB - AS)$	
	$= \frac{1}{2}\,(AB - AC)$	(result in (a))

(d)

$\because$	$\angle AMC = \angle AMS$	(corr. ∠s of cong. Δs)
	$= 90°$	(adj. ∠s on st. line)
and	$\angle ADC = 90°$	(given)
$\therefore$ A, M, D and C are concyclic.		(converse of ∠s in the same seg.)

(e)

	$\angle A'MP = \angle BAP$	(corr. ∠s, $A'M \parallel BA$)
	$= \angle CAP$	(given)
	$= \angle MDP$	(ext. ∠ of cyclic quad.)

(f)

	$\angle MA'P = \angle DA'M$	(common)
	$\angle A'PM = \angle DMA'$	(∠ sum of Δ)
$\therefore$	$\Delta A'MP \sim \Delta A'DM$	(A.A.A.)

Hence, $A'M^2 = A'P \cdot A'D$

$\because$ $A'X = \tfrac{1}{2}\,|AB - AC|$ (result in Q.17)

$= A'M$ (result in (c))

$\therefore$ $A'X^2 = A'P \cdot A'D$.

19. (a) $\because$ $A'X' \cdot A'V = A'X^2$ (result in 12(a))

$= A'P \cdot A'D$ (result in 18(f))

$\therefore$ V, X', P and D are concyclic. (result in 11(b))

(b) Let W be a point on the nine-point circle.

$\angle A'VD = \angle X'VD$ (common)

$= \angle X'PA'$ (ext. ∠ of cyclic quad.)

$= \angle SPB$ (common)

$= \angle ASP - \angle ABC$ (ext. ∠ of Δ)

$= \angle ACB - \angle ABC$ (corr. ∠s of cong. Δs)

$= \angle A'WD$ (result in 7(d))

$\therefore$ A', V, W and D are concyclic. (converse of ∠s in the same seg.)

As A', W, D are points on the nine-point circle, V must also be a point on the nine-point circle.

(c) $\angle UVX' = \angle SX'V$ (base ∠s of isos. Δ)

$= \angle VDP$ (ext. ∠ of cyclic quad.)

$= \angle VDA'$ (common)

$\therefore$ UV is also tangent to the nine-point circle of ΔABC at V.

(converse of ∠ in alt. segment)

The above result shows the nine-point circle always touches the incircle at V.

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